ÉTALE COHOMOLOGY

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Abstract. These are the notes from Emanuel Reinecke's summer mini course on "Étale Cohomology" from May 9-13, 2016. Any and all errors are due to the scribe.

1. Motivation

1.1. **Weil Conjectures.** Given $f \in \mathbf{Z}[x_1, \dots, x_n]$, we are interested in solutions in $\mathbf{Z}$ or $\mathcal{O}_K$. This is very hard, so we first look for solutions in $\mathbf{F}_q$. Recall if $X = \operatorname{Spec} \mathbf{Z}$, the Riemann zeta function is

$$\zeta(X, s) = \prod_{p\colon \text{prime}} \frac{1}{1 - p^{-s}} = \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - |\kappa(x)|^{-s}},$$

where $\kappa(x)$ is the residue field. Similarly, if K is a number field and $X = \operatorname{Spec} \mathcal{O}_K$, we can define the Dedekind zeta function $\zeta(X, s)$.

Definition 1.1. Let X be a finite-type scheme over $\mathbf{Z}$. The *zeta function* of X is

$$\zeta(X, s) = \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - |\kappa(x)|^{-s}}.$$

Exercise 1.2. If X is a variety over $\mathbf{F}_q$, then $\zeta(X, s)$ is called the *Artin zeta function* and it satisfies

$$\zeta(X, s) = \exp\left(\sum_{r=1}^{\infty} \#X(\mathbf{F}_{q^r}) \frac{t^r}{r} \right),$$

where $t = q^{-s}$. This follows from the Taylor expansion of log.

Example 1.3. Let $X = \mathbf{A}^d_{\mathbf{F}_q}$, then $\#X(\mathbf{F}_{q^r}) = q^{rd}$ and

$$\zeta(X, s) = \frac{1}{1 - q^{d-s}}.$$

Example 1.4. Let $X = \mathbf{P}^d_{\mathbf{F}_q} = \mathbf{A}^d_{\mathbf{F}_q} \cup \mathbf{A}^{d-1}_{\mathbf{F}_q} \cup \ldots \cup \mathbf{A}^0_{\mathbf{F}_q}$, then

$$\zeta(X, s) = \frac{1}{(1 - q^{d-s})(1 - q^{d-1-s}) \ldots (1 - q^{-s})}.$$

Example 1.5. Let X be a smooth projective curve over $\mathbf{F}_q$, of genus g. Then,

$$\zeta(X, s) = \frac{P(t)}{(1 - t)(1 - qt)},$$

where $t = q^{-s}$ and $P(t) = \prod_{j=1}^{2g} (1 - \alpha_j t) \in \mathbf{Z}[t]$ with $\alpha_j \in \mathbf{C}$ such that $|\alpha_j| = \sqrt{q}$. The condition $|\alpha_j| = \sqrt{q}$ says that zeros s of $\zeta(X, s)$ have $\operatorname{Re}(s) = 1/2$, hence why this result is called the Riemann hypothesis for curves. It is due to E. Artin for elliptic curves, and to Weil for all curves. This abstract statement gives very concrete results about point counts, as we see below.

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Exercise 1.6. From the above, deduce that $\#X(\mathbf{F}_{q^r}) = 1 + q^r - \left(\sum_{j=1}^{2g} \alpha_j^r\right)$. In particular,

$$|\#X(\mathbf{F}_{q^r}) - (1+q^r)| \leq 2g\sqrt{q^r}.$$

This is called the Weil bound (or when $r = g = 1$, the Hasse bound).

Exercise 1.7. Let $X = \{y^2 z = x^3 - xz^2\} \subset \mathbf{P}^2_{\mathbf{F}_3}$, then:

(1) $\#X(\mathbf{F}_3) = 4$.
(2) Deduce that $\#X(\mathbf{F}_{3^n}) = \begin{cases} 3^n + 1 & n \text{ odd} \\ (3^{n/2} \pm 1)^2 & n \text{ even.} \end{cases}$.

We would like to generalize this. Let X be a smooth projective variety over $\mathbf{F}_q$, of dimension $d = \dim X$. Then, $X(\mathbf{F}_{q^r}) = \mathrm{Fix}(\mathrm{Frob}^r_{X,q})$, the fixed points of the r-th iterate of the Frobenius morphism $\mathrm{Frob}_{X,q} \colon X \to X$. Recall that $\mathrm{Frob}_{X,q} \colon (x_1, \ldots, x_m) \mapsto (x_1^q, \ldots, x_m^q)$.

Question 1.8 (Weil)**.** What if X and $\mathrm{Frob}_{X,q}$ were defined over $\mathbf{C}$?

This is of course ridiculous, but in this case the Lefschetz fixed point theorem says

$$\mathrm{Fix}(\mathrm{Frob}^r_{X,q}) = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}\left(\left(\mathrm{Frob}^r_{X,q}\right)^* \Big| H^i_{sing}(X, \mathbf{Q})\right).$$

Exercise 1.9. From the above, one can use linear algebra to show that

$$\zeta(X,s) = \frac{P_1(t)P_3(t)\ldots P_{2d-1}(t)}{P_0(t)P_2(t)\ldots P_{2d}(t)},$$

where $t = q^{-s}$ and

$$P_i(t) = \det\left(\mathbb{I} - \left(\mathrm{Frob}^r_{X,q}\right)^* t; H^i_{sing}(X,\mathbf{Q})\right),$$

and $\deg P_i(t) = \dim H^i_{sing}(X, \mathbf{Q}) = \beta_i(X)$, the *Betti number*. In addition, one can use Poincaré duality to get a functional equation for ζ.

Doing all of this, Weil made the following conjecture.

Conjecture 1.10 (Weil)**.** Let X be a smooth projective variety over $\mathbf{F}_q$, with $d = \dim X$.

(1) $\zeta(X,s)$ is a rational function in $t = q^{-s}$.
(2) $\zeta(X, d-s) = \pm q^{dE/2} q^{-sE} \zeta(X,s)$, where $E = \Delta^2$ is the self-intersection of the diagonal $\Delta \subset X \times X$.
(3)
$$\zeta(X,s) = \frac{P_1(t)P_3(t)\ldots P_{2d-1}(t)}{P_0(t)P_2(t)\ldots P_{2d}(t)},$$
where $P_0(t) = 1 - t$, $P_{2d}(t) = 1 - q^d t$, and $P_i(t) = \prod_j (1 - \alpha_{ij} t) \in \mathbf{Z}[t]$ with $|\alpha_{ij}| = q^{i/2}$.
(4) If $X = Y \otimes_{\mathcal{O}_K} \mathbf{F}_q$ is the mod p reduction of $Y/\mathcal{O}_K$, for some number field K, then $\deg P_i(t) = \beta_i(Y \otimes \mathbf{C})$.

The problem is, of course, that we cannot use singular cohomology. Instead,

- Why don't we use the Zariski topology? If X is an irreducible scheme and $\mathcal{F}$ is a constant sheaf, then $\mathcal{F}$ is flasque. In particular, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$, so there is no hope of this being true.
- What about de Rham cohomology? If we consider $\mathbb{H}^*(X \otimes \overline{\mathbf{F}_p}, \Omega^*)$, then this only proves the statement over $\overline{\mathbf{F}_p}$, which does not help.

Grothendieck proposed that we need a new "Weil cohomology theory". For example, de Rham cohomology would be a Weil cohomology theory if we were over a field of characteristic zero.

His basic insight was the following: to compute sheaf cohomology, we're more interested in the sheaves on a topological space than the topological space itself.

1.2. **Topological Étale Site.** Let X be a topological space, and let X_{top} be the category of open sets (the objects are open sets and the morphisms are inclusions) together with the data of open coverings (i.e. what are the open covers). We can reformulate the definition of a sheaf on X in the following, more categorical, manner.

Definition 1.11. (1) A *presheaf* of sets is a contravariant functor $\mathcal{F}\colon X_{\text{top}} \to \mathbf{Set}$.
(2) A presheaf $\mathcal{F}$ is a *sheaf* if for any $U \subset S$ open and for any open covering $\{U_i \to U\}$ of U, the following is an equalizer diagram:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

That is, $\mathcal{F}(U)$ consists of exactly the elements of $\prod_i \mathcal{F}(U_i)$ that map to the same element under both maps to $\prod_{i,j} \mathcal{F}(U_i \cap U_j)$. We also rewrite $U_i \cap U_j = U_i \times_U U_j$.
(3) Let $\mathbf{Top}(X)$ be the category of sheaves on the site X_{top}.

This definition makes sense for every *site*, i.e. a category with a good notion of coverings.

Definition 1.12. Let X, Y be topological spaces. A continuous map $f\colon X \to Y$ is *étale* if it is a local homeomorphism; that is, for every point $p \in X$, there is an open $U \ni p$ such that $f(U)$ is open in Y and f is a homeomorphism $U \to f(U)$.

Definition 1.13. The *topological étale site* $X_{\text{ét}}$ of X is given by
(1) The category of étale X-spaces, i.e. the objects are étale maps $U \to X$ and the morphisms are étale morphisms $U \to V$ preserving the structure morphism to X; that is,

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X \end{array}$$

(2) For $U \to X$ étale, open coverings of U are $\left\{ \begin{array}{ccc} U_i & \xrightarrow{f_i} & U \\ & \searrow & \downarrow \\ & & X \end{array} \right\}$ with f_i étale such that $\bigcup_i f_i(U_i) = U$.

Remark 1.14. The categories X_{top} and $X_{\text{ét}}$ are not equivalent, e.g. because objects in $X_{\text{ét}}$ may have automorphisms, but this will not occur in X_{top}.

Definition 1.15. The *étale topos* $\mathbf{\acute{E}t}(X)$ is the category of sheaves on $X_{\text{ét}}$.

Lemma 1.16. $\mathbf{\acute{E}t}(X)$ and $\mathbf{Top}(X)$ are equivalent.

Sketch of proof. Define quasi-inverse functors $\mathbf{\acute{E}t}(X) \xrightarrow{i_*} \mathbf{Top}(X)$ and $\mathbf{Top}(X) \xrightarrow{i^*} \mathbf{\acute{E}t}(X)$ as follows: given $\mathcal{F} \in \mathbf{\acute{E}t}(X)$, define $(i_*\mathcal{F})(U) = \mathcal{F}(U)$. Conversely, given $\mathcal{G} \in \mathbf{Top}(X)$, define

$$(i^*\mathcal{G})\left(U \xrightarrow{h} X\right) = (h^*\mathcal{G})(U).$$

Exercise 1.17. Show that $i^* \circ i_*$ is naturally isomorphic to the identity. (This requires use of the sheaf axiom!)

□

Next time, we will see that the cohomology theories on the two sites are the same. This is the end of the first lecture.

Last time, for a topological space X, we defined the topological étale site $X_{\text{ét}}$ of X, which consists of:
(1) the category of étale X-schemes

(2) for all $U \to X$ étale, the open coverings are $\left\{ \begin{array}{ccc} U_i & \xrightarrow{f_i} & U \\ & \searrow & \downarrow \\ & & X \end{array} \right\}$ with f_i étale and such that $\bigcup_i f_i(U_i) = U$.

We also showed that $\textbf{Ét}(X)$ and $\textbf{Top}(X)$ are equivalent.

Abelian sheaf cohomology is, by definition, the right derived functors of the global sections functor, so in order to show that sheaf cohomology is intrinsically defined for these categories, it suffices to define the global sections functor intrinsically. We will work in the subcategory of abelian sheaves.

On X_{top}, $\mathcal{F}(X) = \text{Hom}_X(h_X, \mathcal{F})$ by the Yoneda Lemma, where $h_X = \text{Hom}_X(-, X)$ is the representable sheaf of continuous maps into X (over X). As $h_X(U) = *$ is a point for any $U \in X_{\text{top}}$, h_X is the final object of $\textbf{Top}(X)$. Similarly, on $X_{\text{ét}}$, the final object of $\textbf{Ét}(x)$ is again h_X. Therefore, in both instances, the global sections functor is just homming out of the final object and moreover the diagram

$$\begin{array}{ccc} \textbf{Top}^{ab}(X) & \xrightarrow{\Gamma} & \textbf{Ab} \\ & \overset{i^*}{\searrow} & \uparrow \Gamma \\ & & \textbf{Ét}^{ab}(X) \end{array}$$

is commutative (and similarly for i_*). Here, $\textbf{Top}^{ab}(X)$ and $\textbf{Ét}^{ab}(X)$ denote the subcategories whose objects are the abelian sheaves. We conclude that one can define Γ and, by extension, sheaf cohomology intrinsically (and compatibly) in terms of the topos.

2. Review of Étale Morphisms

We want an algebraic analogue of the local isomorphisms in complex analytic geometry.

Theorem 2.1 (Implicit Function Theorem)**.** Let x, y be coordinates on $\mathbf{R}^m, \mathbf{R}^n$ respectively. Let $f\colon \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ be C^1 and let $(a, b) \in \mathbf{R}^m \times \mathbf{R}^n$, $c \in \mathbf{R}^n$ with $f(a, b) = c$. If $\left(\frac{\partial f_i}{\partial y_j}(a, b)\right)_{i,j}$ is invertible, then there are $a \in U \subset \mathbf{R}^m$ and $b \in V \subset \mathbf{R}^n$ opens and a C^1-map $g\colon U \to V$ such that

$$\{(x, y) \in \mathbf{R}^m \times \mathbf{R}^n \colon f(x, y) = c\} = \{(x, g(x)) \colon x \in U\}.$$

As a special case, if $c = 0$, then $\{f_1 = f_2 = \ldots = f_n = 0\} \xrightarrow{proj} \mathbf{R}^m$ is (locally) a C^1-diffeomorphism iff $\det \left(\frac{\partial f_i}{\partial y_j}\right)_{i,j} \neq 0$ iff the differential $\left(\frac{\partial f_i}{\partial y_j}\right)_{i,j}$ is an isomorphism of tangent spaces.

This leads to an algebraic definition.

Definition 2.2. A ring map $R \to A$ is called *étale* if there is a presentation $A \simeq R[y_1, \ldots, y_n]/(f_1, \ldots, f_n)$ such that $\det \left(\frac{\partial f_i}{\partial y_j}\right) \in A^\times$.

Definition 2.3. A morphism of schemes $f\colon X \to S$ is *étale* if one of the following equivalent conditions is satisfied:

(1) For any $x \in X$, there is an affine neighbourhood $x \in U = \text{Spec } A \subset X$ and an affine $V = \text{Spec } R \subset S$ such that $f(U) \subset V$ and the induced map $R \to A$ is étale.

(2) For any $x \in X$, there is an affine neighbourhood $x \in U = \text{Spec } A \subset X$ and an affine $V = \text{Spec } R \subset S$ such that $f(U) \subset V$ and the induced map $R \to A$ is *standard étale*, i.e. there is a presentation

$$A \simeq R[y]_g/(h)$$

such that h is monic and $\frac{dh}{dy}$ is invertible in $R[y]_g/(h)$.

(3) f is locally of finite presentation[1] (lofp), flat, and for all $s \in S$,

$$f^{-1}(s) = \bigsqcup_i \text{Spec } k_{i,s},$$

where $\kappa(s) \subset k_{i,s}$ are finite separable extensions (this last condition means f is unramified).

(4) f is lofp, flat, and $\Omega^1_{X/S} = 0$ (this last condition again means f is unramified).

(5) f is lofp and it is formally étale, i.e. for any closed immersion
$$\begin{array}{ccc} \text{Spec } A_0 & \hookrightarrow & \text{Spec } A \\ & \searrow & \downarrow \\ & & S \end{array}$$
with $\ker(A \twoheadrightarrow A_0)$ a square-zero ideal and any diagram as below, there is a unique lift

$$\begin{array}{ccc} \text{Spec } A_0 & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \text{Spec } A & \longrightarrow & S \end{array}$$

Proof. Some directions are obvious (e.g. $(2 \Rightarrow 1)$ and $(3 \Rightarrow 4)$), some are easy (e.g. $(1 \Rightarrow 4)$: if you have a presentation, you can write down the cotangent sheaf), and some are hard (e.g. $(5 \Rightarrow 1)$ - see e.g. "Néron Models" by Bosch, Lütkebohmert, & Raynaud). □

Exercise 2.4. Do as many directions as you can!

Exercise 2.5. This is a special case of $(1 \Rightarrow 5)$. Let A be a finite-type k-algebra and let $B = A[y_1, \ldots, y_n]/(f_1, \ldots, f_n)$ such that $\det\left(\frac{\partial f_i}{\partial y_j}\right) \in B^\times$. Then, Spec $B \to$ Spec A induces an isomorphism of Zariski tangent spaces; that is, for every diagram as below, there is a unique lift

$$\begin{array}{ccc} \text{Spec } k & \longrightarrow & \text{Spec } B \\ \big\downarrow & \nearrow & \big\downarrow \\ \text{Spec } k[\epsilon](\epsilon^2) & \longrightarrow & \text{Spec } A \end{array}$$

Example 2.6.
(1) Let X, S be schemes of finite-type over $\mathbf{C}$, then $f\colon X \to S$ is étale iff $f^{an}\colon X^{an} \to S^{an}$ is a local isomorphism (in the category of analytic spaces i.e. it is a local biholomorphism).
(2) Open immersions are étale. This is important, because we want the étale topology to be finer than the Zariski topology.
(3) If $X \to$ Spec k is étale, then $X = \bigsqcup_i$ Spec K_i, for $k \subset K_i$ separable finite extensions.
(4) If $K \subset L$ is an extension of number fields, then Spec $\mathcal{O}_L \to$ Spec $\mathcal{O}_k$ is étale iff $K \subset L$ is unramified at all finite places.
(5) Take a collection X of $\mathbf{P}^1$'s where each line intersects exactly 2 others. As $\mathbf{P}^1$ is the normalization of the nodal cubic, there is a morphism from X to the nodal cubic, sending each intersection between $\mathbf{P}^1$'s to the node of the cubic. This is an étale morphism (in terms of the picture, it is clearly a local isomorphism!).

Exercise 2.7. Let E be an elliptic curve over $k = \overline{k}$ and $n \in k^\times$, then the "multiplication-by-n" morphism $[n]\colon E \to E$ is a finite étale cover.

[1] A morphism of schemes $f\colon X \to S$ is *locally of finite presentation* (lofp) if for any $x \in X$, there is an open affine neighbourhood $x \in U =$ Spec $A \subset X$ and an affine open $V =$ Spec $R \subset S$ such that $f(U) \subset V$ and such that there is a presentation

$$A \simeq R[y_1, \ldots, y_n]/(f_1, \ldots, f_n).$$

For noetherian schemes, this is the same as being locally of finite-type, but if this is not the case, then this notion is usually better (e.g. it can be defined functorially).

Exercise 2.8 (More challenging)**.** A connected scheme S is *simply connected* if there are no non-trivial finite étale covers. If $k = \overline{k}$, show that

(1) $\mathbf{P}^1_k$ is simply connected (hint: use Riemann-Hurwitz!).
(2) Deduce that $\mathbf{P}^n_k$ is simply connected, for $n \geq 1$.

Proposition 2.9. The following are some permanence properties of étale morphisms.

(1) Étale morphisms are stable under base change.
(2) Étale morphisms are stable under composition.
(3) If $\begin{array}{ccc} U & \xrightarrow{f} & V \\ & \searrow & \downarrow \\ & & S \end{array}$ is a morphism of étale S-schemes, then $f\colon U \to V$ is also étale.

Proof. To prove all of these, we use the characterization in terms of formally étale morphisms. For example, to show (3), we want a unique lift

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} A & \longrightarrow & V \end{array}$$

From the diagram

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} A & \longrightarrow & S \end{array}$$

we get a morphism Spec $A \to U$, and one can show that this is the desired lift from the first diagram. □

3. Étale Topology & Sheaves

Here, we go back to the topological definitions and replace everything with the corresponding algebraic version.

Definition 3.1. The *étale site* $S_{\text{ét}}$ is given by

(1) The category of étale S-spaces, i.e. the objects are étale morphisms $U \to S$ and the morphisms are étale morphisms $U \to V$ preserving the structure morphism to S; that is,

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \\ & & S \end{array}$$

(2) For $U \to S$ étale, open coverings of U are $\left\{ \begin{array}{ccc} U_i & \xrightarrow{f_i} & U \\ & \searrow & \downarrow \\ & & S \end{array} \right\}$ with f_i étale such that $\bigcup_i f_i(U_i) = U$.

This ends the second lecture.

Definition 3.2. (1) An *étale presheaf* of sets (resp. of abelian groups) on S is a contravariant functor $\mathcal{F}\colon \mathrm{S}_{\text{ét}} \to \mathbf{Set}$ (resp. to $\mathbf{Ab}$).

(2) A presheaf $\mathcal{F}$ is a *sheaf* if for all $U \to S$ étale and for all $\left\{ \begin{array}{ccc} U_i & \xrightarrow{f_i} & U \\ & \searrow & \downarrow \\ & & S \end{array} \right\}$ étale coverings,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram.

(3) The *étale topos* $\textbf{Ét}(S)$ is the category of étale sheaves on S. Let $\mathbf{Ab}(S)$ denote the subcategory of abelian sheaves.

Example 3.3. (1) Let A be a set, then $\underline{A}_S \colon U \mapsto A^{\pi_0(U)}$ is the constant sheaf.

(2) Let $\mathbb{G}_a \colon U \mapsto \Gamma(U, \mathcal{O}_U)$, where $\Gamma(U, \mathcal{O}_U)$ are the global sections of the Zariski structure sheaf of U. If we restrict to the Zariski site, then this is just the structure sheaf of S. For this reason, it is sometimes denoted by $\mathcal{O}_S$.

(3) Let $\mathbb{G}_m \colon U \mapsto \Gamma(U, \mathcal{O}_U)^*$, which is sometimes denoted $\mathcal{O}_S^*$.

(4) Let $\mu_n \colon U \mapsto \Gamma(U, \mathcal{O}_U)^*[n]$, where $[n]$ denotes the n-torsion. If n is invertible on S and $\mu_n \subset \mathcal{O}(U)$ (where these are the n-th roots of unity in the usual sense), then $\mu_n \simeq \underline{\mathbf{Z}/n}$ as étale sheaves (though they have different Galois actions).

All of the above examples are incarnations of something more general.

Example 3.4. Given $X \to S$, the presheaf $h_X = \mathrm{Hom}_S(-, X)$ is in fact a sheaf (checking the sheaf axiom uses fpqc descent); sheaves of the form h_X are said to be *representable*. In Example 3.3, the sheaf $\underline{A}_S$ is represented by $\coprod_{a \in A} S$, $\mathbb{G}_a$ is represented by $S \times \mathrm{Spec}\ \mathbf{Z}[t]$, $\mathbb{G}_m$ is represented by $S \times \mathrm{Spec}\ \mathbf{Z}[t, t^{-1}]$, and μ_n is represented by $S \times \mathrm{Spec}\ \mathbf{Z}[t]/(t^n - 1)$.

In some sense, the representable sheaves generate $\textbf{Ét}(S)$: take any sheaf $\mathcal{F} \in \textbf{Ét}(S)$, then it is a coequalizer of a diagram of coproducts of representable sheaves:

$$\coprod_\alpha \mathcal{G}_\alpha \rightrightarrows \coprod_\beta \mathcal{H}_\beta \to \mathcal{F},$$

where $\mathcal{G}_\alpha, \mathcal{H}_\beta$ are representable (and they are represented by not only any S-scheme, but an étale S-scheme!). Thus, we get the étale topos from the étale site by adding certain colimits.

Definition 3.5. (1) An étale sheaf $\mathcal{F}$ is called *locally constant constructible* (lcc) if it is represented by a finite étale S-scheme.

These are exactly the sheaves where S admits an open cover such that, when pulled back to the cover, is the constant sheaf for a finite group on each element of the cover.

(2) An étale sheaf $\mathcal{F}$ is *constructible* if there is a stratification $S = \coprod_i S_i$ into finitely-many pairwise disjoint locally closed subsets S_i such that $\mathcal{F}|_{S_i}$ is lcc.

Theorem 3.6. On a noetherian scheme S, any $\mathcal{F} \in \textbf{Ét}(S)$ is the filtered direct limit of its constructible subsheaves. Similarly, any torsion abelian sheaf $\mathcal{F} \in \mathbf{Ab}(S)$ is the limit of its constructible abelian subsheaves.

This type of theorem enables one to show many of the 'big' statements about étale cohomology: we can first reduce to the case of a noetherian scheme by a limit argument, then make a dévissage to the case of curves, and by the above theorem we reduce to constructible sheaves, from which we reduce to the lcc case, and after passing to a finite cover this is the constant sheaf for a finite group, which we may assume is $\underline{\mathbf{Z}/n}$ by the fundamental theorem of finitely-generated abelian groups. This type of reduction is very common.

Example 3.7. Let $k = k_s$ be a separably-closed field, then any $X \to \mathrm{Spec}\ k$ étale looks like $X = \coprod_{i \in I} \mathrm{Spec}\ k$. Thus, if $\mathcal{F} \in \textbf{Ét}(\mathrm{Spec}\ k)$, $\mathcal{F}(X) = \prod_{i \in I} \mathcal{F}(\mathrm{Spec}\ k)$, so the value of $\mathcal{F}$ on any étale cover is completely determined by the value on $\mathrm{Spec}\ k$. Therefore, the functor $\textbf{Ét}(\mathrm{Spec}\ k) \to \mathbf{Set}$, sending $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}\ k)$, is an equivalence.

Example 3.8. Now let k be an arbitrary field and fix a separable closure k_s, then

$$G = \mathrm{Gal}(k_s/k) = \mathrm{colim}_{k \subset k' \subset k_s} \mathrm{Gal}(k'/k)$$

is a profinite group, where the colimit is taken over $k \subset k' \subset k_s$ finite separable Galois extensions. We can still describe étale sheaves on Spec k in a very nice way: $\mathcal{F} \in \acute{\mathbf{E}}\mathbf{t}(\mathrm{Spec}\ k)$ is determined, as before, by its value on connected covers, but these are just finite separable extensions.

Let $k \subset k' \subset k_s$ be a finite separable extension, then we get $\mathcal{F}(\mathrm{Spec}\ k')$, but there is some compatibility! If $k' \subset k''$ is a finite separable extension, then there is a map $\mathcal{F}(\mathrm{Spec}\ k') \to \mathcal{F}(\mathrm{Spec}\ k'')$. If we restrict to Galois extensions, this fits into the coequalizer diagram

$$\mathcal{F}(\mathrm{Spec}\ k') \to \mathcal{F}(\mathrm{Spec}\ k'') \rightrightarrows \mathcal{F}(\mathrm{Spec}\ k'' \otimes_{k'} k'') = \prod_{g \in \mathrm{Gal}(k''/k')} \mathcal{F}(\mathrm{Spec}\ k'')$$

where, on the factor corresponding to $g \in \mathrm{Gal}(k''/k')$, the two maps are induced by the identity $x \mapsto x$ and $x \mapsto gx$ on k''. As this is a coequalizer diagram, we notice that

$$\mathcal{F}(\mathrm{Spec}\ k') = \mathcal{F}(\mathrm{Spec}\ k'')^{\mathrm{Gal}(k''/k')}.$$

Therefore, we get a directed system and we can take the colimit over finite Galois extensions $k \subset k' \subset k_s$ to obtain

$$M_{\mathcal{F}} := \mathrm{colim}_{k \subset k' \subset k_s} \mathcal{F}(\mathrm{Spec}\ k')$$

and this is a discrete left G-set, i.e. the action $G \times M_{\mathcal{F}} \to M_{\mathcal{F}}$ is continuous, or equivalently, the stabilizer of each element of $M_{\mathcal{F}}$ is an open subset of G.

Theorem 3.9. The functor $\acute{\mathbf{E}}\mathbf{t}(\mathrm{Spec}\ k) \to \{\text{discrete left } G\text{-sets }\}$, sending $\mathcal{F} \mapsto M_{\mathcal{F}}$, is an equivalence of categories.

Exercise 3.10. Fill in the details!

Under this equivalence, $\mathcal{F}(\mathrm{Spec}\ k)$ corresponds to the invariants $M_{\mathcal{F}}^G$, so their right-derived functors coincide; thus, we can compute étale cohomology by computing Galois cohomology!

Exercise 3.11. The functor $(\mathrm{Spec}\ k)_{\text{ét}} \to \acute{\mathbf{E}}\mathbf{t}(\mathrm{Spec}\ k)$, sending $X \mapsto h_X$, is an equivalence of categories. In general, this is just fully faithful.

Example 3.12. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules (on the Zariski site), then we want to make this a sheaf on the étale site. We can define

$$\mathcal{F}_{\text{ét}}\left(U \xrightarrow{h} S\right) = \Gamma(U, h^*\mathcal{F}).$$

This is a presheaf and, though it is not clear that it satisfies the sheaf axiom for étale covers which are not open immersions, it is in fact a sheaf on the étale site.

Exercise 3.13. Show that the sheaf axiom is satisfied in the special case where $\begin{array}{ccc} \mathrm{Spec}\ B & \xrightarrow{f} & \mathrm{Spec}\ A \\ & \searrow & \downarrow \\ & & S \end{array}$, where all morphisms are étale, and $\mathcal{F}|_{\mathrm{Spec}\ A} \simeq \widetilde{M}$. (Hint: first assume that s is a section, and then reduce to this case using the fact that f is faithfully flat.)

Exercise 3.14. Show that the functor $U \mapsto \mathrm{Pic}(U)$ is in general a presheaf, but not a sheaf on the étale site.

4. Operations on Étale Sheaves

4.1. **Sheafification.** Given a presheaf $\mathcal{F}$ on $\mathrm{S}_{\text{ét}}$, one can assign to it a sheaf $\mathcal{F}^+$ with the following universal property: for any $\mathcal{G} \in \textbf{Ét}(S)$ and any morphism $\mathcal{F} \to \mathcal{G}$, there is a unique lift

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

4.2. **Direct & Inverse Images.** Let $f\colon S \to T$ be a morphism.

The direct image is defined for ordinary sheaves $\mathcal{F} \in \mathbf{Zar}(S)$ by the formula

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$

where $U \subset T$ is open. This can be written more categorically: taking preimages is the same as taking fibre products. This leads to the following definition: if $\mathcal{F} \in \textbf{Ét}(S)$, set

$$(f_*\mathcal{F})\,(U \to S) := \mathcal{F}(U \times_T S).$$

If we restrict to $\mathbf{Ab}(S)$, then $f_*\colon \mathbf{Ab}(S) \to \mathbf{Ab}(T)$ is left-exact.

To define inverse images, we could cheat and define f^* to be the unique left-adjoint to f_*. Or, we may construct it similar to how it is done for sheaves on the Zariski site, using a big colimit. Instead, we opt for a less cumbersome construction.

If $\mathcal{F} = h_Z \in \textbf{Ét}(T)$ for some $Z \to T$ étale, then we define $f^*\mathcal{F} := h_{Z\times_T S}$. For $\mathcal{F} \in \textbf{Ét}(T)$ general, there are étale sheaves $\mathcal{G}_\alpha, \mathcal{H}_\beta$ which are representable by étale T-sheaves such that

$$\bigsqcup_\alpha \mathcal{G}_\alpha \rightrightarrows \bigsqcup_\beta \mathcal{H}_\beta \to \mathcal{F}$$

is a coequalizer diagram. Thus, define $f^*\mathcal{F}$ by the coequalizer diagram

$$\bigsqcup_\alpha f^*\mathcal{G}_\alpha \rightrightarrows \bigsqcup_\beta f^*\mathcal{H}_\beta \to f^*\mathcal{F}.$$

As an application, we will discuss the topological invariance of the étale site.

Definition 4.1. A morphism of schemes $f\colon S \to T$ is a *universal homeomorphism* if it is a homeomorphism after every base change, i.e. $f'\colon S \times_T T' \to T'$ is a homeomorphism for all $T' \to T$.

Example 4.2.
(1) Closed immersions defined by nilpotent ideals are universal homeomorphisms.
(2) If $k \subset k'$ is a purely inseparable field extension, then $\operatorname{Spec} k' \to \operatorname{Spec} k$ is a universal homeomorphism.
(3) The normalization $\mathbf{P}^1 \to C$ of the cuspidal cubic C is a universal homeomorphism.

Proposition 4.3. Let $f\colon S \to T$ be a universal homeomorphism.
(1) The functor $T_{\text{ét}} \to \mathrm{S}_{\text{ét}}$, sending $U \to U \times_T S$, gives an isomorphism of étale sites.
(2) The functors $\textbf{Ét}(S) \xrightarrow{f_*} \textbf{Ét}(T)$ and $\textbf{Ét}(T) \xrightarrow{f^*} \textbf{Ét}(S)$ give an equivalence of étale topoi.

This proposition is great because it says that if we want to compute the étale cohomology of e.g. a non-reduced scheme, then we can do so instead for the associated reduced scheme, or we can pass to the perfect closure of the field over which we are working. This ends the third lecture.

Exercise 4.4. If X is an $\mathbf{F}_p$-scheme, then the Frobenius $\mathrm{Frob}_X\colon X \to X$ induces a pullback $\mathrm{Frob}_X^*\colon X_{\text{ét}} \to X_{\text{ét}}$ on étale sites. Show that Frob_X^* is (naturally identified with) the identity functor. (Note: this is stronger than saying this pullback is just some equivalence.)

4.3. **Stalks.**

Definition 4.5. A *geometric point* of a scheme S is a morphism $\overline{s}\colon \operatorname{Spec} k \to S$, where $k = k_s$ is a separably closed field.

One reason why this is the "right notion' of a point is that the higher cohomology of a sheaf on Spec k will vanish when k is separably closed (which is a property that we would like a point to have), but we saw that this may not be the case when k is not separably closed.

Definition 4.6. An *étale neighbourhood* of a geometric point $\overline{s}$ is a diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & U \\ & \overset{\overline{s}}{\searrow} & \downarrow \\ & & S \end{array}$$

with $U \to S$ étale.

In the Zariski topology, in order to look closely at a point s, we consider $\operatorname{Spec} \mathcal{O}_{S,s} = \varprojlim_{s \in U \subset S} U$, where the inverse limit runs over opens $U \subset S$ containing the point s. We make the analogous definition here.

Definition 4.7. The *strict localization* of S at $\overline{s}$ is

$$\lim_{\substack{\operatorname{Spec} k \longrightarrow U \\ \overline{s}\searrow \ \downarrow \\ S}} U = \operatorname{Spec}\ (\operatorname{colim} \Gamma(U, \mathcal{O}_U)) = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{S,\overline{s}},$$

where $\mathcal{O}^{\mathrm{sh}}_{S,s}$ is the *strict henselization* of $\mathcal{O}_{S,\overline{s}}$. This scheme is no longer étale over S, but it is pro-étale.

The strict localization plays the role of a small contractible ϵ-ball around $\overline{s}$, e.g. whenever you have an étale morphism $U \to \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{S,\overline{s}}$, then you can always find a section.

Definition 4.8. The *stalk* of $\mathcal{F} \in \mathbf{\acute{E}t}(S)$ at a geometric point $\overline{s}\colon \operatorname{Spec} k \to S$ is $\operatorname{colim} \mathcal{F}(U)$, where the colimit runs over all étale neighbourhoods U of $\overline{s}$.

Equivalently, $\overline{s}^*\mathcal{F}$ is an étale sheaf over Spec of a separably-closed field, which is completely determined by its global sections. Thus, $\mathcal{F}_{\overline{s}} = (\overline{s}^*\mathcal{F})(\operatorname{Spec} k)$.

Example 4.9. Let $f\colon S \to T$ be a morphism and let $\mathcal{F} \in \mathbf{\acute{E}t}(T)$. For a geometric point $\overline{s}\colon \operatorname{Spec} k \to S$, set $f(\overline{s}) := f \circ \overline{s}$, which is a geometric point of T. Then, $(f^*\mathcal{F})_{\overline{s}} = \mathcal{F}_{f(\overline{s})}$.

Proposition 4.10. A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ on S is a monomorphism/epimorphism/isomorphism iff $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is a monomorphism/epimorphism/isomorphism for all geometric points $\overline{s}$ of S.

5. Computational Methods

Let $\mathbf{Ab}(S)$ denote the subcategory of abelian sheaves inside the topos $\mathbf{\acute{E}t}(S)$ of étale sheaves. This is an abelian category with enough injectives, so the left-exact functor $\Gamma\colon \mathcal{F} \mapsto \mathcal{F}(S)$ has right-derived functors

$$\mathcal{F} \mapsto H^*_{\text{ét}}(S, \mathcal{F}),$$

which are by definition the *étale cohomology groups* of $\mathcal{F}$. We omit the subscript and denote the cohomology groups simply by $H^*(S, \mathcal{F})$.

5.1. **Étale Fundamental Group.** Let X be a path-connected topological space and let M be an abelian group, then recall

$$H^1_{\mathrm{sing}}(X, M) = \mathrm{Hom}(H_{1,\mathrm{sing}}(X, \mathbf{Z}), M) = \mathrm{Hom}(\pi_1(X, x), M)$$

for any basepoint $x \in X$. We would like to construct an analogous group for étale cohomology.

Recall that, in topology, π_1 may be defined via loops or in the following manner: let C, X be connected and locally path-connected and let $C \xrightarrow{f} X$ be a covering map. Let $\mathrm{Aut}(f)$ be the group of deck transformations of the covering map f, i.e. homeomorphisms $\sigma\colon C \to C$ over X.

For any $x \in X$, there is a left-action $\mathrm{Aut}(f) \curvearrowright f^{-1}(x)$, sending a point in the fibre to its image under a deck transformation (which still lies in the fibre because a deck transformation is a map *over* X). Unique path-lifting implies that the action is free, and we say that the cover is regular if the action is transitive.

There is a second action on fibres: according to unique path-lifting, for any $x \in X$ we get a right-action $f^{-1}(x) \curvearrowleft \pi_1(X, x)$, called the monodromy action. If f is regular, then

$$\pi_1(X, x)/f_*\pi_1(C, c) \simeq \mathrm{Aut}(f).$$

In particular, if f is the universal cover, then $\pi_1(X, x) \simeq \mathrm{Aut}(f)$. Thus, we get a way to compute π_1 by looking at deck transformations of the universal cover, and this is what we would like to generalize to the algebraic world.

The problem is that universal covers seldom exist in the algebraic world! The solution is to use a limit of finite covers instead, recovering the profinite completion of π_1. Now we just need a dictionary from the topological to the algebraic world.

A choice of basepoint corresponds to a choice of geometric point (in future, we assume implicitly that the basepoints in a finite étale cover have the same residue field). A finite cover corresponds to a finite étale cover, and unique path-lifting corresponds to the Rigidity Lemma, explained below.

Lemma 5.1 (Rigidity Lemma)**.** Let $\begin{array}{ccc} X & \overset{f,g}{\rightrightarrows} & Y \\ & \searrow & \downarrow \\ & & S \end{array}$ be two morphisms, from a connected S-scheme X to a separated étale S-scheme Y. Say $f(\overline{x}) = g(\overline{x}) \in Y(k)$ for some geometric point $\overline{x}$ of X with residue field k, then $f = g$.

Proof. The morphism $Y \to S$ is étale, hence unramified, so the diagonal $\Delta\colon Y \to Y \times_S Y$ is an open immersion. However, $Y \to S$ is also separated, so Δ is a closed immersion as well, which means Δ is a connected component of $Y \times_S Y$, i.e. we may write $Y \times_S Y = \Delta \sqcup Z$ for some other S-scheme Z.

Now, $\mathrm{im}(f \times g\colon X \to Y \times_S Y) \cap \Delta \neq \emptyset$ by assumption and X is connected, so $\mathrm{im}(f \times g) \subseteq \Delta$. Therefore, the 2 functions coincide at every point. $\square$

A regular cover in topology will correspond to a Galois cover: let $f\colon X \to S$ be a finite étale cover of degree n, then the Rigidity Lemma implies that $|\mathrm{Aut}(X/S)| \leq n$.

Definition 5.2. We say that f is *Galois* if $|\mathrm{Aut}(X/S)| = n$. In this case, the *Galois group* of f is $\mathrm{Gal}(X/S) := \mathrm{Aut}(X/S)^{\mathrm{op}}$, where op denotes opposite group (this is done because in the topological world, we have a left and a right action on fibres).

Example 5.3. Let $K \subset L$ be a finite extension of fields. It is Galois iff Spec $L \to$ Spec K is a Galois cover. This example justifies the name.

Now we are ready to define the analogue of (the profinite completion of) π_1.

Definition 5.4. Let $(S, \overline{s})$ be a pointed connected scheme, then the étale fundamental group is defined to be

$$\pi_1(X, s) := \varprojlim_{(X,\overline{x}) \to (S,\overline{s})} \mathrm{Aut}(X/S)^{\mathrm{op}} = \varprojlim_{(X,\overline{x}) \to (S,\overline{s})} \mathrm{Gal}(X/S),$$

where the inverse limit runs over all connected finite Galois covers $(X,\overline{x}) \to (S,\overline{s})$. The fundamental group is covariant in $(S,\overline{s})$, so we get pushforwards.

Example 5.5. (1) Let X be of finite-type over $\mathbf{C}$, then $\pi_1^{\text{ét}}(X,\overline{x}) = \pi_1^{\text{top}}(X(\mathbf{C}),x)^\wedge$, where $^\wedge$ denotes the profinite completion. This also says that every topological cover comes from an étale (in particular, algebraic) cover, which is something like the Riemann Existence Theorem - so this is not a trivial statement!

(2) Let $S = \operatorname{Spec} k$ and $\overline{s}$ a choice of separable closure k_s, then $\pi_1(S,\overline{s}) = \operatorname{Gal}(k_s/k)$.
(3) $\pi_1(\operatorname{Spec} \mathbf{Z}) = 1$ because there are no finite extensions of $\mathbf{Q}$ which are unramified at every finite place.
(4) If $k = \overline{k}$, then $\pi_1(\mathbf{P}^n_k) = 1$, due to Exercise 2.8.
(5) As the cuspidal cubic is universally homeomorphic to $\mathbf{P}^1$, the fundamental group of the cuspidal cubic is isomorphic to $\pi_1(\mathbf{P}^1) = 1$.

Exercise 5.6. Compute π_1 of Spec $\mathbf{Z}[\frac{1}{n}]$, Spec $\mathbf{Z}_{(p)}$, and of the nodal cubic (hint: see Example 2.6.5).

Exercise 5.7 (Harder)**.** Let $k = \overline{k}$ and let E be an elliptic curve over k. Show (algebraically) that

$$\pi_1(E) = \begin{cases} \prod_{\ell \text{ prime}} \mathbf{Z}_\ell \times \mathbf{Z}_\ell & \operatorname{char} k = 0, \\ \prod_{\ell \neq p} \mathbf{Z}_\ell \times \mathbf{Z}_\ell & \operatorname{char} k = p \text{ and } \operatorname{Hasse}(E) = 0, \\ \mathbf{Z}_p \times \prod_{\ell \neq p} \mathbf{Z}_\ell \times \mathbf{Z}_\ell & \operatorname{char} k = p \text{ and } \operatorname{Hasse}(E) \neq 0. \end{cases}$$

Theorem 5.8 (Grothendieck)**.** Let $(S,\overline{s})$ be a pointed connected scheme. There is an equivalence of categories, functorial in $(S,\overline{s})$, of

$$\{\text{finite étale covers}\} \xrightarrow{\simeq} \{\text{finite discrete left } \pi_1(S,\overline{s})\text{-sets}\},$$

sending $X \to S$ to $X(\overline{s})$.

This vastly generalizes Example 3.8 and Theorem 3.9 in the case of lcc sheaves.

Exercise 5.9. Fill in the details. In particular, how does $\pi_1(S,\overline{s})$ act on $X(\overline{s})$? It is obvious in the case when $X \to S$ is Galois; otherwise, choose $Y \to X \to S$ such that $Y \to S$ is Galois, then $Y \to X$ is necessarily Galois.

5.2. **Čech Theory.** This works basically the same way as for the Zariski topology (this will even work for a general site). Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be an étale cover of X. We first define Čech cohomology for an abelian presheaf $\mathcal{G} \in \mathbf{PAb}(X)$. Define the *Čech complex* to be

$$\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{G}) = \left(\prod_{i\in I} \mathcal{G}(U_i) \to \prod_{i,j\in I} \mathcal{G}(U_i \times_X U_j) \to \prod_{i,j,k\in I} \mathcal{G}(U_i \times_X U_j \times_X U_k) \to \dots \right).$$

It is important to include the $i = j$ case in the products (this is often omitted when working with the Zariski topology). The p-th (étale) Čech cohomology is by definition the p-th cohomology of this complex; that is,

$$\check{H}^p(\mathcal{U},\mathcal{G}) := H^p(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{G})).$$

If $\mathcal{G} \in \mathbf{Ab}(X)$ is a sheaf, then $\check{H}^0(\mathcal{U},\mathcal{G}) = \Gamma(X,\mathcal{G})$, but this is not necessarily true if $\mathcal{G}$ is just a presheaf - we really need the sheaf axiom. To compare Čech cohomology to étale cohomology, we must construct the Čech-to-derived functor spectral sequence. The idea is to use the Grothendieck spectral sequence on the diagram

$$\begin{array}{ccc} \mathbf{Ab}(X) & \xrightarrow{F} & \mathbf{PAb}(X) \\ & {\scriptstyle\Gamma}\searrow & \downarrow {\scriptstyle \check{H}^0(\mathcal{U},-)} \\ & & \mathbf{Ab} \end{array}$$

where F is the forgetful functor.

Exercise 5.10. In order to use the Grothendieck spectral sequence, we must check the following.

(1) Show that F is left-exact and $R^iF(\mathcal{G}) = \mathcal{H}^i(\mathcal{G})$, which is the presheaf $U \mapsto H^i(U, \mathcal{G})$.
(2) Show that $\check{H}^0(\mathcal{U}, -)$ is left-exact and $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$.
(3) Show that F applied to an injective object of $\mathbf{Ab}(X)$ is an $\check{H}^0(\mathcal{U}, -)$-acyclic object of $\mathbf{PAb}(X)$.

The Grothendieck spectral sequence gives

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{G})) \Rightarrow H^{p+q}(X, \mathcal{G}) \text{ for any } \mathcal{G} \in \mathbf{Ab}(X).$$

Define $\check{H}^p(X, \mathcal{G})$ to be the colimit of $\check{H}^p(\mathcal{U}, \mathcal{G})$ over all refinements of $\mathcal{U}$ (the maps between Čech cohomology groups are independent of the choice of the maps between refinements). The spectral sequences are compatible with the refinement maps, so we get another spectral sequence

$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(\mathcal{G})) \Rightarrow H^{p+q}(X, \mathcal{G}) \text{ for any } \mathcal{G} \in \mathbf{Ab}(X).$$

Theorem 5.11. For all $\mathcal{G} \in \mathbf{Ab}(X)$, the natural map $\check{H}^1(X, \mathcal{G}) \xrightarrow{\simeq} H^1(X, \mathcal{G})$ is an isomorphism.

Proof. From the abutment of $E_2^{p,q}$, we get an exact sequence

$$0 \to \check{H}^1(X, \mathcal{G}) \to H^1(X, \mathcal{G}) \to \check{H}^0(X, \mathcal{H}^1(\mathcal{G})),$$

so it is enough to show $\check{H}^0(X, \mathcal{H}^1(\mathcal{G})) = 0$. For every element $\xi \in H^1(U, \mathcal{G})$ for some $U \to X$ étale, there is an étale cover $\{U_i \to U\}$ such that $\xi|_{U_i} = 0$ for all i. In particular, every element of every $H^1(U, \mathcal{G})$ vanishes in the colimit over $U \to X$. (This is the "locality of cohomology".)

□

This ends the fourth lecture.

The goal from last time was to compute H^1 from π_1. In order to do this, we need a geometric notion.

Definition 5.12. Let X be a scheme and let $\mathcal{G} \in \mathbf{\acute{E}t}(X)$ be a sheaf of groups. Then, $\mathcal{F} \in \mathbf{\acute{E}t}(X)$ together with a right $\mathcal{G}$-action is called a *right $\mathcal{G}$-torsor* if

(1) $\mathcal{F}_{\overline{x}} \neq \emptyset$ for every geometric point $\overline{x}$.
(2) The morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \times \mathcal{F}$, sending $(s, g) \mapsto (s, sg)$, is an isomorphism.

Similarly, one may define left $\mathcal{G}$-torsors.

One should think of $\mathcal{G}$-torsors as the analogue of principal homeogeneous space for $\mathcal{G}$, and of the second condition as saying that $\mathcal{G}$ acts freely and transitively on fibres.

Example 5.13. Let L/K be a finite Galois extension and let $G = \mathrm{Gal}(L/K)$, then the constant sheaf $\underline{G}$ gives a sheaf of groups in $\mathbf{\acute{E}t}(\mathrm{Spec}\ K)$. Recall that $\mathrm{Spec}\ L \to \mathrm{Spec}\ K$ is étale, and we claim that $\mathcal{F} = h_{\mathrm{Spec}\ L}$ is a right $\underline{G}$-torsor.

The condition (1) is clear. The condition (2) is saying (by the Yoneda Lemma) that the map $L \otimes_K L \to \prod_{g \in G} L$, sending $x \otimes y \mapsto x \cdot g(y)$, is an isomorphism, which is true for any Galois extension.

Exercise 5.14. Show that, more generally, for any connected Galois cover $Y \to X$ with Galois group G, h_Y is a right $\underline{G}$-torsor.

Remark 5.15. If $\mathcal{F}$ is a $\mathcal{G}$-torsor and $s \in \mathcal{F}(U)$ is a local section, then $\mathcal{G}|_U \to \mathcal{F}|_U$, given by $g \mapsto sg$, is an isomorphism. In particular, $\mathcal{G}$ and $\mathcal{F}$ are (étale) locally isomorphic.

Furthermore, $\mathcal{G}$ is lcc iff it is (étale) locally a constant finite sheaf iff $\mathcal{F}$ is lcc iff $\mathcal{F} = h_Y$ for some finite étale cover $Y \to X$.

Example 5.16. There is a correspondence

$$\{(\text{right}) \ \mathrm{GL}(n)\text{-torsors}\} \leftrightarrow \{(\text{étale}) \text{ locally free sheaves of rank-}n\},$$

where a (right) $\mathrm{GL}(n)$-torsor $\mathcal{F}$ is sent to the locally free sheaf $\mathcal{F} \times^{\mathrm{GL}(n)} \mathcal{O}_X^n$, which is the product $\mathcal{F} \times \mathcal{O}_X^n$ modulo the equivalence relation $(fg, s) \sim (f, gs)$ for g a section of $\mathrm{GL}(n)$. Conversely, send a locally free sheaf $\mathcal{V}$ to the torsor $\mathcal{I}som(\mathcal{V}, \mathcal{O}_X^n)$.

Furthermore, using descent theory, one can show that there is a correspondence between étale-locally free sheaves of rank-n and Zariski-locally free sheaves of rank-n. In particular, there is a correspondence

$$\{(\text{right})\ \mathrm{GL}(1) = \mathbb{G}_m\text{-torsors }\}/\text{isom's} \leftrightarrow \mathrm{Pic}(X).$$

The former can be endowed with a group structure so that this is an isomorphism of groups.

If $\mathcal{G} \in \mathbf{Ab}(X)$, then one obtains a natural commutative group structure on the collection of (right) $\mathcal{G}$-torsors, up to isomorphism: given two $\mathcal{G}$-torsors $\mathcal{F}, \mathcal{F}'$, the product $\mathcal{F} \times \mathcal{F}'$ is a $\mathcal{G} \times \mathcal{G}$-torsor and we define $\mathcal{F} \diamond \mathcal{F}' := \mathcal{F} \times \mathcal{F}'/\Delta^-$, where $\Delta^- = \{(g, -g) \colon g \in \mathcal{G}\}$ is the antidiagonal of $\mathcal{G} \times \mathcal{G}$.

This can be described in another way: $\mathcal{F} \diamond \mathcal{F}'$ is the unique $\mathcal{G}$-torsor such that there is map $\mathcal{F} \times \mathcal{F}' \xrightarrow{f} \mathcal{F} \diamond \mathcal{F}'$ which is equivariant with respect to $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$; that is,

$$f\left((g, h) \cdot (s, t)\right) = (g + h) \cdot f(s, t).$$

Exercise 5.17. There is a group isomorphism $\{(\text{étale})\ \mathbb{G}_m\text{-torsors}\}/\text{isom's} \simeq \mathrm{Pic}(X)$.

Exercise 5.18. For any $\mathcal{G} \in \mathbf{Ab}(X)$, $\check{H}^1(X, \mathcal{G}) \simeq \{\text{right } \mathcal{G}\text{-torsors}\}/\text{isom's}$.

Corollary 5.19. For any $\mathcal{G} \in \mathbf{Ab}(X)$, $H^1(X, \mathcal{G}) \simeq \{\text{right } \mathcal{G}\text{-torsors}\}/\text{isom's}$.

Corollary 5.20 (Hilbert Satz 90)**.** Let $K \subset L$ be a Galois extension and $G = \mathrm{Gal}(L/K)$. Then, $H^1_{\mathrm{Gal}}(G, L^*) = 0$.

Proof. The inflation-restriction sequence gives an injection $H^1_{\mathrm{Gal}}(G, L^*) \hookrightarrow H^1_{\mathrm{Gal}}(G_K, \overline{K}^*)$, where $G_K = \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K. As G_K-modules correspond to sheaves on Spec K, it follows that

$$H^1_{\mathrm{Gal}}(G_K, \overline{K}^*) \simeq H^1(\mathrm{Spec}\ K, \mathbb{G}_m) \simeq \mathrm{Pic}(\mathrm{Spec}\ K) = 0.$$

□

This leads us to the big theorem.

Theorem 5.21. Let $(X, \overline{x})$ be a connected pointed scheme and let G be a finite abelian group. Then,

$$H^1(X, \underline{G}) \simeq \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), G),$$

where G is equipped with the discrete topology.

Proof. We saw that

$$H^1(X, \underline{G}) \simeq \{\text{right } \underline{G}\text{-torsors}\}/\text{isom's},$$

and, since $\underline{G}$ is lcc, each right $\underline{G}$-torsor is represented by a finite étale cover, so the above is isomorphic to

$$\left\{\begin{matrix}\text{finite étale covers } Y \to X \text{ with a right } G\text{-action} \\ \text{such that } Y \times G \xrightarrow{\simeq} Y \times Y, \text{ sending } (y, g) \mapsto (y, gy), \text{ is an isomorphism}\end{matrix}\right\}/\text{isom's}.$$

The condition that the map $Y \times G \to Y \times Y$ be an isomorphism says that G acts freely and transitively on fibres, so by Theorem 5.8, the above is isomorphic to

$$\left\{\begin{matrix}\text{finite (discrete) sets with compatible left } \pi_1(X, \overline{x})\text{-action} \\ \text{and free and transitive right } G\text{-action}\end{matrix}\right\}/\text{isom's}.$$

As G acts freely and transitively on a set, it can be identified with the set; thus, we really just have π_1 acting on G, which is completely determined by where π_1 sends the identity of G. It follows that the above is isomorphic to the group $\mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), G)$. □

Remark 5.22. The result may still hold when G is infinite (and still equipped with the discrete topology), but we need some assumptions on X (e.g. if X is normal and noetherian).

Example 5.23. (1) If $k = \overline{k}$, then $H^1(\mathbf{P}^n_k, \underline{\mathbf{Z}/n}) = 0$, which is what is expected from singular cohomology!

(2) Let X be connected, separated, and of finite-type over $\mathbf{C}$, then

$$H^1(X, \underline{\mathbf{Z}/n}) \simeq \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbf{Z}/n) \simeq \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{top}}(X(\mathbf{C}), x)^\wedge, \mathbf{Z}/n).$$

However, continuous homomorphisms out of a profinite completion and into a profinite group are the same as homomorphisms out of the original group (into the same profinite group), so

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{top}}(X(\mathbf{C}), x)^\wedge, \mathbf{Z}/n) \simeq \mathrm{Hom}(\pi_1(X(\mathbf{C}), x), \mathbf{Z}/n) \simeq H^1_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Z}/n).$$

Example 5.24. What happens if we don't take finite coefficients? Let X be normal and connected, then

$$H^1(X, \mathbf{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), \mathbf{Z}) = 0,$$

since $\pi_1(X, \overline{x})$ is a profinite, in particular compact, group and so any continuous group homomorphism to an infinite discrete group must be trivial. The same would occur if we had used $\mathbf{Z}_\ell$-coefficients. The reason for this is that in algebraic geometry we can only detect connected components and finite irreducible covers, but not infinite irreducible covers.

Exercise 5.25. Consider the étale cover of the nodal cubic C given by an infinite chain of $\mathbf{P}^1$'s. Show that it is a nontrivial $\mathbf{Z}$-torsor over C, thus $H^1(C, \mathbf{Z}) \neq 0$. However, the argument of Example 5.24 still holds, so $\mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), \mathbf{Z}) = 0$. Conclude that, if a scheme X is not normal and the group G is infinite, then the conclusion of Theorem 5.21 may not hold.

5.3. **Cohomology of Quasicoherent Sheaves.** For a proper scheme of finite-type over $\mathbf{C}$, GAGA says that sheaf cohomology on the Zariski site and on the analytic site coincide for coherent sheaves, so we may expect the same in this setting.

Recall that, given a sheaf on X_{Zar}, we may construct a sheaf on $X_{\text{ét}}$: there is a functor $\mathbf{Zar}(X) \xrightarrow{i^*} \text{\'{E}t}(X)$, where $i^*\mathcal{F}$ is the sheafification of the presheaf

$$\left(U \xrightarrow{h} X\right) \mapsto (h^*\mathcal{F})(U).$$

There is an adjoint functor $\text{\'{E}t}(X) \xrightarrow{i_*} \mathbf{Zar}(X)$, just given by restricting the site. There are global sections functors Γ from both topoi to $\mathbf{Ab}$. A δ-functor argument shows that we get a natural "Zariski-étale comparison" morphism

$$\theta \colon H^*(X_{\mathrm{Zar}}, \mathcal{F}) \to H^*(X_{\text{ét}}, i^*\mathcal{F}).$$

For simplicity, we will write $\mathcal{F}$ for $i^*\mathcal{F}$.

Theorem 5.26. If $\mathcal{F}$ is quasicoherent, then θ is an isomorphism, i.e. $H^i_{\mathrm{Zar}}(X, \mathcal{F}) \simeq H^i_{\text{ét}}(X, \mathcal{F})$.

Proof. The idea is to use the Čech-to-derived functor spectral sequence on an open affine cover $\mathcal{U}$. In fact, we get 2 different spectral sequences - one for the Zariski topology and one for the étale topology:

$$\begin{array}{ccc} \check{H}^p_{\mathrm{Zar}}(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) & \Longrightarrow & H^{p+q}_{\mathrm{Zar}}(X, \mathcal{F}) \\ \downarrow \theta & & \downarrow \theta \\ \check{H}^p_{\text{ét}}(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) & \Longrightarrow & H^{p+q}_{\text{ét}}(X, \mathcal{F}) \end{array}$$

It suffices to show that the map θ on the E_2-page is an isomorphism for all p, q. Unraveling the definition of Čech cohomology, it suffices to prove the statement when X is the intersection of finitely-many affines; in particular, for X separated. Repeat this argument when X is separated to reduce to the case when X is the intersection of finitely-many affines inside a separated scheme, hence is affine. Therefore, it suffices to show the statement when X is affine.

It is always true that $H^0_{\mathrm{Zar}}(X, \mathcal{F}) = H^0_{\text{ét}}(X, \mathcal{F})$, by definition. Thus, it suffices to show the statement for $i > 0$. By Serre's criterion, we know that $H^i_{\mathrm{Zar}}(X, \mathcal{F}) = 0$, because $\mathcal{F}$ is quasicoherent; it remains to show that

$H^i_{\text{ét}}(X, \mathcal{F}) = 0$. This is a direct computation using Čech cohomology. It boils down to showing the exactness (in positive degrees) of the sequence

$$B \otimes_A M \to B \otimes_A B \otimes_A M \to B \otimes_A B \otimes_A B \otimes_A M \to \dots,$$

where $A \to B$ is faithfully flat and M is an A-module. $\square$

This theorem allows us to compute a lot of étale cohomology groups!

5.4. **The Kummer & Artin-Schreier Sequences.** Let S be a scheme.

Exercise 5.27 (Kummer sequence)**.** If n is invertible on S, then the following sequence is exact:

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0,$$

where $\mathbb{G}_m \to \mathbb{G}_m$ is given by $t \mapsto t^n$. Left-exactness is clear, but right-exactness is not (in fact, it is false for the Zariski topology). Hint: pass to an étale cover on which a given section of $\mathbb{G}_m$ has an n-th root.

The Kummer sequence plays the role of the exponential exact sequence for analytic sheaf cohomology (though there is no formal relation between the two).

Exercise 5.28 (Artin-Schreier sequence)**.** If $p = 0$ on S, then the following sequence is exact:

$$0 \to \underline{\mathbf{Z}/p} \to \mathbb{G}_a \to \mathbb{G}_a \to 0,$$

wehre $\mathbb{G}_a \to \mathbb{G}_a$ is given by $t \mapsto t^p - t$. Once again, this is not true in the Zariski topology.

The Artin-Schreier sequence plays the role of the Poincaré lemma in the analytic setting, in the sense that it provides a resolution of $\mathbf{Z}/p$ by coherent sheaves. Once again, there is no way to make this relationship formal.

Using this, one can show (with quite a bit of work) the following.

Theorem 5.29 (M. Artin)**.** Let X be separated, finite-type over $\mathbf{C}$, then $H^i_{\text{ét}}(X, \underline{\mathbf{Z}/n}) \simeq H^i_{\text{sing}}(X(\mathbf{C}), \mathbf{Z}/n)$.

5.5. **Back to the beginning.** Recall that we wanted a nice Weil cohomology theory with characteristic zero coefficients in order to tackle the Weil conjectures. The torsion coefficients $\mathbf{Z}/n$ are not enough. To get a Weil cohomology theory, we have to consider the *ℓ-adic cohomology*, which is defined to be

$$H^i(X, \mathbf{Z}_\ell) := \varprojlim_n H^i_{\text{ét}}(X, \mathbf{Z}/\ell^n).$$

This is different from $H^i_{\text{ét}}(X, \mathbf{Z}_\ell)$, which is zero provided that X is normal. Note that cohomology does not commute with inverse limits of sheaves.

Example 5.30.

$$H^1(X, \mathbf{Z}_\ell) = \varprojlim_n \operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbf{Z}/\ell^n) = \operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbf{Z}_\ell),$$

where the last group consists of continuous group homomorphisms from $\pi_1^{\text{ét}}(X, \overline{x})$ to $\mathbf{Z}_\ell$, which is now equipped with its *profinite topology*!