Multiplier Ideals

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Abstract

Multiplier ideals are an important tool in complex algebraic/analytic geometry. They are often used to study e.g. the singularities of divisors, or metrics on holomorphic line bundles. In this mini course, I will give an overview of the analytic and algebraic theories, along with plenty of examples. Topics will include:

1. Analytic multiplier ideals;
2. Review of **Q**-divisors and log resolutions;
3. Algebraic multiplier ideals: basic properties, and examples;
4. Positivity and vanishing theorems;
5. Metrics on line bundles and their multiplier ideals;
6. Zariski decompositions and the Fujita Approximation theorem.

Lecture 1

References.

- [Algebraic] §§9–10 in: Robert Lazarsfeld. *Positivity in algebraic geometry. II.* Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics] 49. Berlin: Springer-Verlag, 2004. ISBN: 3-540-22534-X. DOI: 10.1007/978-3-642-18808-4. MR: 2095472.
- [Analytic] Jean-Pierre Demailly. "Multiplier ideal sheaves and analytic methods in algebraic geometry." *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000).* ICTP Lect. Notes 6. Trieste: Abdus Salam Int. Cent. Theoret. Phys., 2001, pp. 1–148. MR: 1919457.

Plan.

- [Monday–Wednesday] General theory;
- [Thursday–Friday] "Fancy" analytic theory and applications, in particular
  - Kollár's results on singularities of θ-divisors on principally polarized abelian varieties and
  - the Fujita approximation theorem.

  Note that there is a notion of an asymptotic multiplier ideal that can be used to derive these results purely algebraically, but we will instead be using the analytic theory for multiplier ideals.

1 Analytic multiplier ideals

Let X be a complex manifold, with $\mathcal{O}_X$ the sheaf of holomorphic functions. Consider a function

$$\varphi\colon X \to \mathbf{R} \cup \{-\infty\}.$$

Associated to this function, we define a sheaf of ideals $\mathscr{J}(\varphi)$ by

$$\mathscr{J}(\varphi)_x = \left\{ f \in \mathcal{O}_{X,x} : |f| e^{-\varphi} \in L^2_{\mathsf{loc}}(x) \right\}$$

that is, the stalks of $\mathscr{J}(\varphi)$ at $x \in X$ consist of germs of holomorphic functions f such that

$$\int_{\text{nbhd of } x} |f|^2 e^{-2\varphi}\, d\lambda < +\infty,$$

*Notes were taken by Takumi Murayama, who is responsible for any and all errors.

where $d\lambda$ is the Lebesgue measure. This might seem nonstandard to algebraists, but we will see later that $|f|e^{-\varphi}$ is some local version of a metric on a line bundle over X.

Definition 1.1. $\mathscr{J}(\varphi)$ is the *multiplier ideal sheaf* associated to φ.

With appropriate assumptions on the function φ, we have the following (deep) result:

Theorem 1.2 (Nadel)**.** *If φ is plurisubharmonic (psh), then the multiplier ideal sheaf $\mathscr{J}(\varphi)$ is a coherent sheaf of ideals.*

We won't discuss the general definition for plurisubharmonic functions, but we will mainly be concerned with the following

Main Examples 1.3 (of plurisubharmonic functions)**.** Functions of the form $\varphi = c \cdot \log|f|$, where $c > 0$ is a constant real number and f is a holomorphic function, are plurisubharmonic (psh). We will also consider finite $\mathbf{R}_{>0}$-linear combinations of functions of this form, and maxima taken over finite families of such functions.

1.1 Examples

Example 1.4. Let $X = \mathbf{C}$, and consider $\varphi = c \cdot \log|z|$, $c > 0$. We want to know for which $c > 0$, the multiplier ideal ideal is trivial, that is, $\mathscr{J}(\varphi) = \mathcal{O}_X$.

Since any function $f \in \mathcal{O}_{X,x}$ is bounded in a neighborhood of 0, if $\mathscr{J}(\varphi) = \mathcal{O}_X$, then it suffices to ask when the integral

$$\int_{\text{nbhd of }0} |1|^2 e^{-2\varphi}\, d\lambda$$

is finite. By substituting our particular function φ, this is equivalent to asking when

$$\int_{B(0,\epsilon)} \frac{1}{|z|^{2c}}\, d\lambda = 2\pi \int_0^\epsilon \frac{dr}{r^{2c-1}} < +\infty,$$

where the second integral uses polar coordinates. But this holds if and only if $c < 1$.

When $c \geq 1$, we have $f \in \mathscr{J}(\varphi)$ if and only if

$$\int \frac{|f|^2}{|z|^{2c}}\, d\lambda < +\infty$$

where from now on, an integral without a subscript will be taken over a ball around singularities of the integrand, i.e., zeroes of the denominator. This integral is finite if and only if $z^{\lfloor c \rfloor} \mid f$, and so if $c \geq 1$, we have that $\mathscr{J}(\varphi) = \left(z^{\lfloor c \rfloor}\right)$.

Example 1.5. Let $X = \mathbf{C}^n$, and $\varphi = \sum_{j=1}^n c_j \cdot \log|z_j|$, $c_j > 0$. We want to know when $\mathscr{J}(\varphi) = \mathcal{O}_X$ again, i.e., when the integral

$$\int \frac{1}{\prod_{j=1}^n |z_j|^{2c_j}}\, d\lambda$$

is finite. By Fubini's theorem, this is equivalent to asking when

$$\int \frac{1}{|z_j|^{2c_j}}\, d\lambda < +\infty$$

for all j, which holds if and only if $c_j < 1$ for all j.

As in the previous example, if $c_j \geq 1$ for some j, then

$$\mathscr{J}(\varphi) = \left(z_1^{\lfloor c_1 \rfloor} \cdots z_n^{\lfloor c_n \rfloor}\right).$$

This example is useful since multiplier ideals are defined locally, and so on any manifold, we are often in a similar situation:

Example 1.6. Let X be a complex manifold, $\varphi = \sum_{j=1}^{n} c_j \cdot \log|g_j|$ for $c_j > 0$, where g_j are holomorphic functions such that $D_j = g_j^{-1}(0)$ are irreducible and have simple normal crossings (snc). This says that locally we're in the situation of Example 1.5. Then, $f \in \mathscr{J}(\varphi)$ if and only if

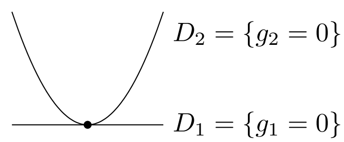
$$\int \frac{|f|^2}{\prod |g_j|^{2c_j}}\, d\lambda < +\infty,$$

which holds if and only if $g_j^{\lfloor c_j \rfloor} \mid f$ for all j. Thus,

$$\mathscr{J}(\varphi) = \left(\prod_{j=1}^{n} g_j^{\lfloor c_j \rfloor} \right).$$

Warning 1.7. If the D_j's do not intersect transversely, this is false!

Exercise 1.8. Let $X = \mathbf{C}^2$, $g_1 = z$, $g_2 = w - z^2$. In this situation the D_j do not intersect transversely:



Check that $z \in \mathscr{J}\big(\log|z(w - z^2)|\big)$ but $z \notin (z(w - z^2))$.

This Example 1.6 can be phrased quite algebraically: let $D := \sum_{j=1}^{n} c_j D_j$ be an "**R**-divisor." Then,

$$\mathscr{J}(\varphi) = \mathcal{O}_X\Big(-\sum \lfloor c_j \rfloor D_j \Big) =: \mathcal{O}_X(-\lfloor D \rfloor).$$

This gives an indication as to how we will define multiplier ideals in the algebraic setting.

When we are not in the nice situation of Example 1.6, we will want to make a change of coordinates so that we *are* in this situation.

Example 1.9. Let $X = \mathbf{C}^2$, $\varphi = c \cdot \log|w^2 - z^3|$, $c > 0$. Again, we want to know when $\mathscr{J}(\varphi) = \mathcal{O}_X$.

We need to know when

$$\int_{\text{nbhd of } 0} \frac{1}{|w^2 - z^3|^{2c}}\, d\lambda < +\infty.$$

Make the following change of coordinates: $z = uv^2$, $w = uv^3$. Then,

$$w^2 - z^3 = (uv^3)^2 - (uv^2)^3 = u^2 v^6 (1 - u),$$

and the Jacobian is

$$d\lambda(z, w) = |uv^4|^2\, d\lambda(u, v).$$

So we want to know when

$$\int \frac{d\lambda}{|w^2 - z^3|^{2c}} = \int_{\text{nbhd of } 0} \frac{|uv^4|^2}{|u^2 v^6|^{2c} |1 - u|}\, d\lambda < +\infty.$$

But note that $|1 - u|$ is bounded in a neighborhood of the origin, so we only need to know when

$$\int \frac{1}{|u|^{4c-2}} \cdot \frac{1}{|v|^{12c-8}}\, d\lambda < +\infty.$$

But this holds if and only if $4c - 2 < 2$ and $12c - 8 < 2$, i.e., when both $c < 1$ and $c < 5/6$. Thus, $\mathscr{J}(\varphi) = \mathcal{O}_X$ if and only if $c < 5/6$.

When $c \geq 5/6$, it's a bit complicated to describe $\mathscr{J}(\varphi)$. We will revisit this example later from an algebraic perspective.

Note that, in this Example 1.9, there are some subtleties: you need to be careful which "neighborhoods of zero" you are integrating over, and why a neighborhood of zero after the change of coordinates has the same convergence behavior as before the change of coordinates. But in this case, the difference between a preimage of a neighborhood of zero in the old coordinates and a neighborhood in the new coordinates has a convergent integral for all c, and so it does not affect the value of the integral.

Goal 1.10. We want to define multiplier ideals algebraically for a given divisor D. So far, we know what we should get when D is a simple normal crossings divisor. We therefore want an analogue of the analytic change of coordinates we used above, along with its Jacobian, and we want to be able to bundle these two pieces of information together in a nice way.

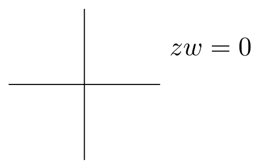
1.2 Application to Singularities

Consider $f \in \mathbf{C}[z_1, \ldots, z_n]$, or more generally $f \in \mathbf{C}\{z_1, \ldots, z_n\}$, the ring of germs of holomorphic functions at 0, such that $f(0) = 0$. We want to understand the singularity of f at 0.

The most naïve measure of the singularity of f at 0 is $\operatorname{ord}_0(f)$, the *order of vanishing* of the function f at 0, but this is not the best way to study singularities:

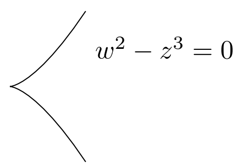
Example 1.11. The following singularities have the same order of vanishing:

1. The node $zw = 0$:



   has $\operatorname{ord}_0(zw) = 2$;

2. The cusp $w^2 - z^3 = 0$:



   has $\operatorname{ord}_0(w^2 - z^3) = 2$.

We want to define something that distinguishes the two singularities above, and we also want this object to detect that somehow, a cusp singularity is worse than a node singularity.

We therefore define the following, more subtle invariant:

Definition 1.12. The *complex singularity exponent* of f at $0 \in \mathbf{C}^n$ is

$$c_0(f) = \sup\left\{c > 0 : \mathscr{J}(c \cdot \log|f|)_0 = \mathcal{O}_{\mathbf{C}^n,0}\right\} = \sup\left\{c > 0 : \frac{1}{|f|^{2c}} \text{ locally integrable near } 0\right\}.$$

This is also called the *log canonical threshold* in algebraic geometry.

We note that we don't really need to use multiplier ideals here, but this shows the multiplier ideal gives more information in general than $c_0(f)$.

Example 1.13. In Example 1.11,

1. The node $zw = 0$ has $c_0(zw) = 1$;
2. The cusp $w^2 - z^3 = 0$ has $c_0(w^2 - z^3) = 5/6$.

The idea is that a smaller c_0 should reflect "worse" singularities of f at 0.

2 Preliminaries

Our next goal is to discuss what replace changes of coordinates and Jacobians in the algebraic setting.

For all of our lectures, a *variety* will be a separated integral scheme of finite type over $\mathbf{C}$, while we allow a *subvariety* to not necessarily be irreducible.

2.1 Divisors

Definition 2.1. If X is normal, then a $\mathbf{Q}$*-divisor* is a formal $\mathbf{Q}$-linear combination

$$D = \sum a_i D_i, \quad a_i \in \mathbf{Q}$$

where D_i are codimension 1 irreducible subvarieties. We say D is *effective* if $a_i \geq 0$, and *integral* if $a_i \in \mathbf{Z}$.

The *round* of D is

$$\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i.$$

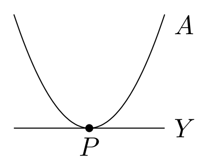
We will also use square brackets instead of $\lfloor \cdot \rfloor$ to denote the round.

We say D is $\mathbf{Q}$*-Cartier* if mD is Cartier and integral for some $m \in \mathbf{Z}$.

Note that when X is smooth, every $\mathbf{Q}$-divisor is $\mathbf{Q}$-Cartier.

Warning 2.2. Rounding does not behave well with algebraic operations, for example pullbacks:

Let $X = \mathbf{C}^2$, Y an axis, and A a parabola intersecting the axis, as below:



Then, letting $D = \frac{1}{2}A$, we have $D|_Y = \frac{1}{2}2P = P$, so $\lfloor D|_Y \rfloor = P$, while $\lfloor D \rfloor|_Y = 0$. This issue will not appear if we have simple normal crossings (snc) divisors.

Goal 2.3. We want to define the multiplier ideal of an effective $\mathbf{Q}$-divisor D on X, so that if D is snc, then $\mathscr{J}(D) = \mathcal{O}_X(-\lfloor D \rfloor)$.

Note that in general, you can still define multiplier ideals without assumptions on effectivity, in which case you would have to use fractional ideal sheaves instead of regular ones. Instead, for simplicity, we will only consider the effective case.

2.2 Log Resolutions

When D is not snc, we need a "change of coordinates" in order to "monomialize" it, i.e., in order to get something snc. These are what log resolutions will do, replacing changes of coordinates that we used before.

Definition 2.4. Let $D = \sum_i a_i D_i$ be a $\mathbf{Q}$-divisor on X, a normal variety. A *log resolution* of D is a projective birational map $\mu \colon X' \to X$ such that X' is smooth, and $\mu^* D + \operatorname{exc}(\mu)$ is an snc divisor. Here, $\operatorname{exc}(\mu)$ is the *exceptional divisor*, i.e., the locus where μ fails to be an isomorphism (note that we assume that this locus has codimension 1). Log resolutions are also called *embedded resolutions*.

Remark 2.5. It is not true in general that the exceptional locus of a birational morphism is a divisor; see

Karl Schwede. *What is the definition of exceptional divisor?* MathOverflow. VERSION: 2012-04-05. URL: `http://mathoverflow.net/q/93219`.

We also note that $\mu^* D$ only obviously makes sense if D is $\mathbf{Q}$-Cartier. The following paper discusses how to make sense of this for the purposes of defining multiplier ideals for arbitrary $\mathbf{Q}$-divisors D:

Tommaso de Fernex and Christopher D. Hacon. "Singularities on normal varieties." *Compos. Math.* 145.2 (2009), pp. 393–414. ISSN: 0010-437X. DOI: `10.1112/S0010437X09003996`. MR: 2501423.

We often assume X is smooth so that these issues do not arise.

Example 2.6. Let $X = \mathbf{C}^2$, and let D be the cuspidal cubic $w^2 - z^3 = 0$. In this case, the log resolution looks like the following:

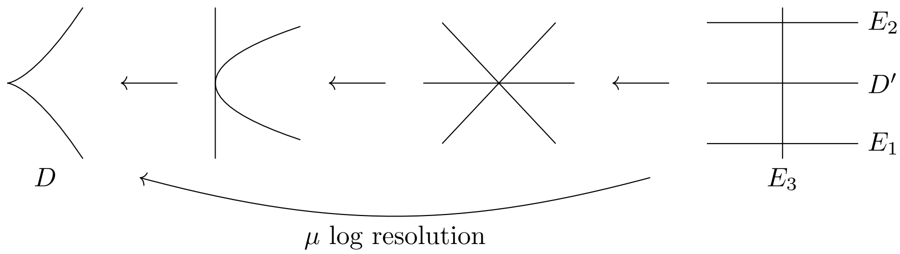


Figure 2.7: Log resolution for the cupisdal cubic.

Exercise 2.8. Compute this example, and check that

$$\mu^* D = 2E_1 + 3E_2 + 6E_3 + D', \qquad \operatorname{exc}(\mu) = E_1 + 2E_2 + 4E_3.$$

Remark 2.9. When we did our calculation for the cuspidal cubic before, the change of coordinates from Ex. 1.9 was not a log resolution, for example since it was not a proper map: $\mu^{-1}(0,0) = \{uv = 0\}$ is not compact (in the analytic topology).

Lecture 2

We have not yet said why log resolutions exist in general:

Theorem 2.10 (Hironaka)**.** *If X is a smooth variety, and D is an effective $\mathbf{Q}$-divisor, then there exists a log resolution $\mu\colon X' \to X$ of D. Moreover, we can find μ such that it is a sequence of blowups at smooth centers.*

The idea of this theorem is that we can always "monomialize" a given divisor.

Lastly, we need to discuss what the Jacobian is in our algebraic context.

Definition 2.11. Let X be a smooth variety (so that K_X is $\mathbf{Q}$-Cartier). If $\mu\colon X' \to X$ is a log resolution, then the *relative canonical divisor* is

$$K_{X'/X} := \operatorname{div}(\det \operatorname{Jac}(\mu))\,.$$

Note that this is supported on the exceptional divisor $\operatorname{exc}(\mu)$.

Equivalently, we can define $K_{X'/X}$ to be the unique effective divisor such that its support is $\operatorname{exc}(\mu)$, and

$$K_{X'/X} \equiv_{\text{lin}} K_{X'} - \mu^* K_X.$$

The first definition is more useful for computations. They are equivalent by a local computation.

Exercise 2.12. Let X be a smooth variety, with Y a smooth subvariety of codimension $r \geq 2$. Consider the blowup $\pi\colon \tilde{X} = \mathrm{B}\ell_Y\, X \to X$. Then,

$$K_{\tilde{X}/X} \equiv_{\text{lin}} K_{\tilde{X}} - \pi^* K_X = (r-1)E,$$

where E is the exceptional divisor of the blowup. [Hint: $\operatorname{Pic}(\tilde{X}) = \pi^* \operatorname{Pic}(X) \oplus \mathbf{Z} \cdot E$.]

It will be useful to know how the relative canonical divisor pushes forward:

Proposition 2.13. *If X is a smooth variety, and $\mu\colon X' \to X$ is a birational morphism, then*
1. $\mu_* \mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X$;
2. $\mu_* \mathcal{O}_{X'}(K_{X'}) = \mathcal{O}_X(K_X)$.

Proof. If μ can be factored into a sequence of blowups, then you can use Exercise 2.12, in which case (1) follows from (2) by the projection formula.

In the general case, you can proceed by noting $K_{X'/X}$ is effective, and so (1) is automatic. (2) then follows by the projection formula. $\square$

3 Algebraic Multiplier Ideals

Fix X a smooth variety.

Definition 3.1. Let D be an effective **Q**-divisor, and let $\mu\colon X' \to X$ be a log resolution of D. Then, the *multiplier ideal of D* is

$$\mathscr{J}(X,D) := \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor).$$

We also denote this by $\mathscr{J}(D)$ when it is clear what the ambient variety X is.

Remarks 3.2.

- If D is not effective, then we can still define the multiplier ideal in this way, except $\mathscr{J}(D)$ may only be a *fractional* ideal sheaf.
- You can also define multiplier ideals for ideal sheaves or linear series.

Proposition 3.3. *$\mathscr{J}(D)$ is independent of the log resolution μ.*

Proof. **Step 1.** Any two log resolutions of D can be dominated by a third: Consider the fibre product

$$\begin{tikzcd}
 & X' \times_X X'' \arrow[dl] \arrow[dr] & X''' \arrow[l] \\
X' \arrow[dr, "\mu"'] & & X'' \arrow[dl, "\nu"] \\
 & X &
\end{tikzcd}$$

where $X''' \to X' \times_X X''$ is a log resolution of $(\mu^* D + \mathrm{exc}(\mu)) + (\nu^* D + \mathrm{exc}(\mu))$.

It therefore suffices to show the proposition for $X'' \xrightarrow{\nu} X' \xrightarrow{\mu} X$ where μ and $\mu \circ \nu$ are log resolutions of D.

Step 2. Check that

$$\mathcal{O}_{X'}(-\lfloor \mu^* D \rfloor) = \nu_* \mathcal{O}_{X''}(K_{X''/X'} - \lfloor \nu^* \mu^* D \rfloor).$$

Step 3. Compute:

$$\begin{aligned}
\mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor) &= \mu_* \left(\mathcal{O}_{X'}(K_{X'/X}) \otimes \nu_* \mathcal{O}_{X''}(K_{X''/X'} - \lfloor \nu^* \mu^* D \rfloor) \right) \\
&= \mu_* \nu_* \left(\nu^* \mathcal{O}_{X'}(K_{X'/X}) \otimes \mathcal{O}_{X''}(K_{X''/X'} - \lfloor \nu^* \mu^* D \rfloor) \right) \\
&= (\mu \circ \nu)_* \mathcal{O}_{X''}(K_{X''/X} - \lfloor \nu^* \mu^* D \rfloor)
\end{aligned}$$

where the penultimate equality is by the projection formula, and the last equality uses the "chain rule" for relative canonical divisors:

$$K_{X''/X'} + \nu^* K_{X'/X} = K_{X''} - \nu^* K_{X'} + \nu^* K_{X'} - \nu^* \mu^* K_X = K_{X''/X},$$

where equality in this line denotes linear equivalence. $\square$

Example 3.4. Let $X = \mathbf{C}^2$, and let $D = \{y^2 - x^3 = 0\}$. We saw a log resolution of D, $\mu\colon X' \to X$ (see Figure 2.7). Recall that

$$\mu^* D = 2E_1 + 3E_2 + 6E_2 + D', \qquad K_{X'/X} = E_1 + 2E_2 + 4E_3.$$

Now consider

$$\mathscr{J}(X, t \cdot D) = \mu_* \mathcal{O}_{X'}\left((1 - \lfloor 2t \rfloor)E_1 + (2 - \lfloor 3t \rfloor)E_2 + (4 - \lfloor 6t \rfloor)E_3 - \lfloor t \rfloor D'\right).$$

1. When $0 \le t < 5/6$, we have that each coefficient is positive, and so $\mathscr{J}(t \cdot D) = \mathcal{O}_X$.
2. When $5/6 \le t < 1$, $\mathscr{J}(t \cdot D) = \mu_* \mathcal{O}_{X'}(-E_3)$, where $\mathcal{O}_{X'}(-E_3)$ consists of those functions h such that $\mu^* h$ vanishes on E_3. But $\mu_* E_3$ is the origin, and so we obtain $\mathscr{J}(t \cdot D) = (x, y)$.
3. When $t \ge 1$, $\mathscr{J}(t \cdot D) = (y^2 - x^3)\mathscr{J}((t-1)D)$ (in general, you can pull out an integral divisor, and so only the fractional part matters).

In general, there is a discrete set of rational numbers tending to $+\infty$ such that the multiplier ideal changes; these are called "jumping numbers."

3.1 Basic Properties

Fact 3.5. Our algebraic and analytic constructions coincide: Given a divisor $D = \sum a_i D_i$, then locally

$$D = \Big\{ \prod_i g_i^{a_i} = 0 \Big\}, \quad \text{where} \quad D_i = \{g_i = 0\}.$$

Let $\varphi_D = \log \big| \prod_i g_i^{a_i} \big|$. Then, we claim the following:

Claim 3.6. $\mathscr{J}(D)^{\mathrm{an}} = \mathscr{J}(\varphi_D)$.

The only real work required is to show the analytic definition is independent of choices of local equations and of log resolutions, which follows because any two local equations differ by a nonvanishing function.

Proposition 3.7. *If A is an integral divisor, then $\mathscr{J}(A) = \mathcal{O}_X(-A)$.*

Proof. Let $\mu\colon X' \to X$ be a log resolution of A. Then, $\lfloor \mu^* A \rfloor = \mu^* A$. In this case, we can use the projection formula to see that

$$\mathscr{J}(A) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \mu^* A) = \mathcal{O}_X(-A).$$ □

Proposition 3.8 (“Multiplier ideals are invariant under small perturbations”)**.** *If D, D' are effective $\mathbf{Q}$-divisors, then $\mathscr{J}(D) = \mathscr{J}(D + \epsilon \cdot D')$ for all $0 < \epsilon \ll 1$.*

Proof. $\lfloor \mu^* D \rfloor = \lfloor \mu^*(D + \epsilon D') \rfloor$. □

Remark 3.9 (Kollár–Bertini)*.* You can make this work for all $0 < \epsilon < 1$, so long as D' is general in some base-point-free linear system.

Proposition 3.10. *If D is an effective $\mathbf{Q}$-divisor, and A is an integral divisor, then*

$$\mathscr{J}(D + A) = \mathscr{J}(D) \otimes \mathcal{O}_X(-A).$$

Proof. Let $\mu\colon X' \to X$ be a log resolution of $D + A$. Then,

$$\mathscr{J}(D + A) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor - \mu^* A) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor) \otimes \mathcal{O}_X(-A),$$

where the second equality is by the projection formula. □

Proposition 3.11. *If D, D' are effective $\mathbf{Q}$-divisors, and $D \le D'$, then $\mathscr{J}(D') \subseteq \mathscr{J}(D)$.*

Proof. If $\mu\colon X' \to X$ is a log resolution of D', then the claim follows since $\mathcal{O}_{X'}(-\lfloor \mu^* D \rfloor) \subseteq \mathcal{O}_{X'}(-\lfloor \mu^* D \rfloor)$. □

Lecture 3

We now want to examine how multiplier ideals change under birational transformations.

Proposition 3.12. *Let $f\colon Y \to X$ be a birational morphism, and let D be an effective $\mathbf{Q}$-divisor on X. Then,*

$$\mathscr{J}(X, D) = f_*\big(\mathscr{J}(Y, f^* D) \otimes \mathcal{O}_Y(K_{Y/X})\big).$$

More symmetrically,

$$\mathcal{O}_X(K_X) \otimes \mathscr{J}(X, D) = f_*(\mathscr{J}(Y, f^* D) \otimes \mathcal{O}_Y(K_Y)).$$

We think of this as a “chain rule for multiplier ideals.”

Proof. We take a log resolution $\nu\colon Y' \to Y$ of $f^* D + \operatorname{exc}(f)$, in which case $f \circ \nu = \mu$ is also a log resolution of D. We want to compute both sides of our claimed equality. To do so, recall the “chain rule” for relative canonical divisors:

$$K_{Y'/X} = K_{Y'} - \mu^* K_X = K_{Y'} - \nu^* K_Y + \nu^* K_Y - \mu^* K_X = K_{Y'/X} - \nu^* K_{Y/X},$$

where as before, equality denotes linear equivalence. Now compute:

$$\begin{aligned} \mathscr{J}(X, D) &= \mu_* \mathcal{O}_{Y'}(K_{Y'/X} - \lfloor \mu^* D \rfloor) \\ &= f_* \nu_* \big(\mathcal{O}_{Y'}(K_{Y'/Y} - \lfloor \nu^* f^* D \rfloor)\big) \\ &= f_*\big(\mathscr{J}(Y, f^* D) \otimes \mathcal{O}_Y(K_{Y/X})\big) \end{aligned}$$

using the projection formula in the last line. □

We also have some results which say when the multiplier ideal is not trivial. Recall that if D is integral and effective, and $\operatorname{mult}_x D \geq 1$ (i.e., $x \in D$), then $\mathscr{J}(D) = \mathcal{O}_X(-D) \subseteq \mathfrak{m}_x$. We can generalize this property in the following way for higher order multiplicities:

Proposition 3.13. *Let X be an n-dimensional variety, D an effective $\mathbf{Q}$-divisor on X. If $\operatorname{mult}_x D \geq n+p-1$ where $p \geq 1$, then $\mathscr{J}(D) \subseteq \mathfrak{m}_x^p$; in particular, $\mathscr{J}(D)$ is nontrivial.*

Proof. Let $\mu\colon X' \to X$ be a log resolution of D, such that it dominates the blowup at x, i.e., we have the following factorization:

$$\begin{tikzcd}
& \mu\colon X' \arrow[rr] \arrow[dr] & & X \\
E \arrow[ur, hook] & & \mathrm{B}\ell_x X \arrow[ur] &
\end{tikzcd}$$

Then, the (pullback of the) exceptional divisor E of the blowup satisfies the following properties:

- $\operatorname{ord}_E(K_{X'/X}) = n-1$;
- $\operatorname{ord}_E(\mu^* D) = \operatorname{mult}_x D$;
- $\mathcal{O}_{X'}(-mE) = \mathfrak{m}_x^m$, $m \geq 1$.

This gives:

$$\operatorname{ord}_E(K_{X'/X} - \lfloor \mu^* D \rfloor) = (n-1) - \operatorname{mult}_x D \leq -p.$$

This is really saying that

$$\mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor) \subseteq \mathcal{O}_{X'}(-pE),$$

and pushing this forward to X, we obtain

$$\mathscr{J}(D) \subseteq \mu_* \mathcal{O}_{X'}(-pE) = \mathfrak{m}_x^p. \qquad \square$$

This concludes our list of basic properties for multiplier ideals.

4 Positivity and Vanishing Theorems

We now want to introduce some notions of positivity for divisors, and how they give us vanishing theorems for cohomology of multiplier ideals.

We fix X a smooth, projective variety of dimension n.

4.1 Positivity

Recall that if A is an ample divisor on X, then it has two important properties:

1. [Nakai–Moishezon] For every irreducible subvariety $V \subset X$ with $\dim V > 0$, we have $(A^{\dim V} \cdot V) > 0$.
2. [Growth of sections] For $k \gg 0$, the dimension of the space of global sections of kA grows as

$$h^0(X, kA) \underset{\substack{\text{Serre}\\\text{vanishing}}}{\sim} \chi(X, kA) \underset{\substack{\text{Asymptotic}\\\text{Riemann–Roch}}}{\sim} \frac{(A^n)}{n!} k^n.$$

We also have the following important vanishing theorem for adjoint line bundles of ample divisors:

Theorem 4.1 (Kodaira)**.** *Let A be an ample divisor on X. Then, $H^i(X, \mathcal{O}_X(K_X + A)) = 0$ for $i > 0$.*

We will generalize amplitude in different ways based on our properties above:

Definition 4.2. Let D be a $\mathbf{Q}$-divisor on X.

1. D is *nef* if for all irreducible subvarieties $V \subset X$ of $\dim V > 0$, we have $(D^{\dim V} \cdot V) \geq 0$.
2. D is *big* if the *volume* of D, defined as

$$\operatorname{vol}(D) := \limsup_{k \to +\infty} \frac{h^0(X, kD)}{k^n/n!}$$

is positive.

For example, if D is ample, then $\mathrm{vol}(D) = (D^n) > 0$, so ample implies big.

For divisors which are both big and nef, we have an analogue of the Kodaira vanishing theorem:

Theorem 4.3 (Kawamata–Viehweg)**.** *Let D be an integral divisor, E an snc $\mathbf{Q}$-divisor, such that $D - E$ is big and nef. Then, $H^i(X, \mathcal{O}_X(K_X + D - \lfloor E \rfloor)) = 0$.*

Note that if X is not projective, we have to be a bit careful about what we mean by a big divisor or a nef divisor (because the analytic definitions may be different).

4.2 Vanishing Theorems

We want to use multiplier ideals to get rid of the snc hypothesis on E in Theorem 4.3.

Theorem 4.4 (Local Vanishing)**.** *Let X be a smooth variety, D an effective $\mathbf{Q}$-divisor, and $\mu\colon X' \to X$ a log resolution of D. Then, $R^j \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor) = 0$ for $j > 0$.*

We use the following useful Lemma:

Lemma 4.5. *Let $f\colon Y \to Z$ be a morphism between projective varieties, and let A be an ample divisor on Z. Suppose $\mathscr{F} \in \mathsf{Coh}(Y)$ is such that $H^i(Y, \mathscr{F} \otimes \mathcal{O}_Y(f^* mA)) = 0$ for all $m \gg 0$ and $j > 0$. Then, $R^j f_* \mathscr{F} = 0$ for all $j > 0$.*

Proof Idea. Use the Leray spectral sequence. □

Proof of Local Vanishing. For simplicity, assume both X and X' are projective. Pick A an ample divisor on X such that $A - D$ is also ample. Then, $\mu^*(A - D)$ is both big and nef. In this case, we can use the Kawamata–Viehweg theorem to conclude

$$H^j(X', \mathcal{O}_{X'}(K_{X'} + \mu^* A - \lfloor \mu^* D \rfloor)) = 0, \quad \text{for } j > 0.$$

Replacing A with mA (for all $m \geq 1$), the Lemma implies that $R^j \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor) = 0$. □

The reduction to the projective case is a bit subtle; look at Lazarsfeld's book for details.

Theorem 4.6 (Nadel)**.** *Let X be a smooth projective variety, and let D be an integral divisor. Let E be an effective $\mathbf{Q}$-divisor such that $D - E$ is big and nef. Then, $H^i(X, \mathcal{O}_X(K_X + D) \otimes \mathscr{J}(E)) = 0$ for $i > 0$.*

Remark 4.7. If you replace X with a compact Kähler manifold, and $\mathscr{J}(E)$ with $\mathscr{J}(\varphi)$ for φ plurisubharmonic, then you still get the same vanishing. Esnault and Viehweg had this result in the algebraic setting in the 1980s, but it did not gain popularity until Nadel gave an analytic proof in 1990, which he used to find obstructions to the existence of Kähler–Einstein metrics (or more precisely, an obstruction to the continuity method working).

Sketch of Proof. Let $\mu\colon X' \to X$ be a log resolution of E. Local vanishing implies that

$$R^j \mu_* \big(\mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* E \rfloor) \otimes \mu^* \mathcal{O}_X(K_X + D)\big) = R^j \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* E \rfloor) \otimes \mathcal{O}_X(K_X + D) = 0.$$

Now use the Leray spectral sequence; for simplicity, we just show $H^1 = 0$.

On the E_2 page, we have the first diagonal:

$$0 \longrightarrow H^1(X, R^0 \mu_* \mathscr{F}) \longrightarrow H^{1+0}(X', \mathscr{F}) \longrightarrow H^0(X, R^1 \mu_* \mathscr{F})$$

where $\mathscr{F} = \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* E \rfloor) \otimes \mu^* \mathcal{O}_X(K_X + D)$. By local vanishing above, we have $R^1 \mu_* \mathscr{F} = 0$, and so $H^1(X, R^0 \mu_* \mathscr{F}) \xrightarrow{\sim} H^1(X', \mathscr{F})$, which we write in terms of multiplier ideals:

$$H^1(X, \mathcal{O}_X(K_X + D) \otimes \mathscr{J}(E)) \cong H^1(X, \mathscr{F}) = 0$$

where this vanishes because of the Kawamata–Viehweg theorem. □

4.3 Singularities of Theta Divisors

We setup Kollár's result.

Definition/Theorem 4.8. *Let X be a smooth variety, and let D be an effective $\mathbf{Q}$-divisor. We say that the pair (X, D) is log-canonical (lc) if one of the following equivalent conditions holds:*

- [Multiplier Ideals] $\mathscr{J}(X,(1-\epsilon)D) = \mathcal{O}_X$ *for all* $0 < \epsilon < 1$.
- [Integrability] *If* $D_{\mathrm{loc}} = \left\{\prod_j g_j^{c_j} = 0\right\}$, *then* $\left(\prod_j g_j^{c_j}\right)^{-1} \in L^1_{\mathsf{loc}}$.
- [Log Discrepancies] $a_E(X, D) \geq -1$ *for all divisors E over X.*

Recall that if $f\colon Y \to X$ is a birational morphism, then the *log discrepancy* is defined as the coefficient in front of E in the decomposition $K_Y - f^*(K_X + D) = \sum_E a_E(X, D) \cdot E$, where E are prime.

Note in this second criterion that we use L^1 instead of L^2. In the L^2 case, we have that $\mathscr{J}(D) = \mathcal{O}_X$ if and only if $\left(\prod_j g_j^{c_j}\right)^{-1} \in L^2_{\mathsf{loc}}$, which corresponds to the klt condition in birational geometry.

Examples 4.9.

- If $\dim X = 2$, then D lc if and only if normal crossings (this means you use the étale or analytic topology, as opposed to snc which is in the Zariski topology).
- If $\dim X = 3$, then you get more examples: pinch points like $D = \{x^2 - y^2 z = 0\} \subset X = \mathbf{C}^3$, simple elliptic points, some cusps.

Recall that a *principally polarized abelian variety (ppav)* is a pair (A, θ) where A is an abelian variety, and θ is an ample divisor with $h^0(A, \theta) = 1$.

Example 4.10. Jacobians of curves are ppav's.

Theorem 4.11 (Andreotti–Mayer 1967)**.** *The θ-divisor of a generic ppav is smooth.*

Theorem 4.12 (Kollár 1995)**.** *The θ-divisor is lc.*

Lecture 4

This Theorem has a simple Corollary:

Corollary 4.13. *If* $\dim A = g$, *then* $\operatorname{mult}_x \theta \leq g$ *for all* $x \in \theta$.

Proof. Use Proposition 3.13. □

Lemma 4.14. *If (A, θ) is a ppav, and $Z \subset A$ is a nonempty subset, then* $H^0(Z, \mathcal{O}_Z(\theta)) \neq 0$.

Proof Sketch. Let $a \in A$, and $\theta_a := \theta + a$ a "translate" of θ, that is, $\{x + a : x \in \theta\}$ (you can realize this as pulling back θ by the isomorphism induced by multiplication by a^{-1}). For a general choice $a \in A$, we have $Z \not\subset \theta_a$, and so $h^0(Z, \mathcal{O}_Z(\theta_a)) > 0$. Sending $a \to 0$, we see that the dimension of $H^0(Z, \mathcal{O}_Z(\theta_a))$ can only jump up by the semicontinuity theorem, and so $h^0(Z, \mathcal{O}_Z(\theta)) > 0$. □

Proof of Theorem. Assume there exists $\epsilon > 0$ such that $\mathscr{J}((1-\epsilon)\theta) \neq \mathcal{O}_X$. Define Z to be the closed subscheme defined by $\mathscr{J}((1-\epsilon)\theta)$. Then, $Z \subset \theta$, since $(1-\epsilon)\theta \leq \theta$, and using Proposition 3.11. Now look at the short exact sequence for closed subschemes, twisted by θ:

$$0 \longrightarrow \mathcal{O}_A(\theta) \otimes \mathscr{J}((1-\epsilon)\theta) \longrightarrow \mathcal{O}_A(\theta) \longrightarrow \mathcal{O}_Z(\theta) \longrightarrow 0,$$

and since $K_A \equiv_{\mathrm{lin}} 0$ for an abelian variety, we have by Nadel vanishing that $H^1(A, \mathcal{O}_A(\theta) \otimes \mathscr{J}((1-\epsilon)\theta)) = 0$, and so $H^0(A, \mathcal{O}_A(\theta)) \overset{\rho}{\twoheadrightarrow} H^0(Z, \mathcal{O}_Z(\theta))$ is surjective. The unique (up to scaling) section in $H^0(A, \mathcal{O}_A(\theta))$ must vanish along θ, hence also vanishes along Z since $Z \subset \theta$. Thus, its image under ρ in $H^0(Z, \mathcal{O}_Z(\theta))$ is zero, and so $\rho \equiv 0$. Thus, $H^0(Z, \mathcal{O}_Z(\theta)) = 0$, which contradicts Lemma 4.14. □

Remark 4.15 (Ein–Lazarsfeld 1997).

- For all $m \geq 1$, and $E \in |m\theta|$, the *pluri-theta divisor* $\frac{1}{m}E$ is log canonical.
- The θ-divisor is normal and has rational singularities.

This concludes our discussion about applications of algebraic multiplier ideals.

5 Metrics on Line Bundles

We now move onto the "fancy analytic story." To begin, we start with some preliminaries about metrics on line bundles. Fix X a complex manifold, and L a holomorphic line bundle on X.

Definition 5.1. A *metric* on L is a real-valued function $\|\cdot\|\colon L \to \mathbf{R}_{\geq 0}$ such that for all $x \in X$, and all $\xi \in L_x \cong \mathbf{C}$, we have that

- $\|\lambda \cdot \xi\| = |\lambda| \cdot \|\xi\|$ for all $\lambda \in \mathbf{C}$;
- $\|\xi\| = 0$ if and only if $\xi = 0$.

It is convenient to write the metric additively: $\varphi = -\log\|\cdot\|$.

Recall that giving a section s of L is equivalent to giving local sections $s_i \in \mathcal{O}_X(U_i)$ where (U_i) is a trivializing open cover for L with transition maps g_{ij}, such that $s_i = g_{ij}s_j$ on $U_i \cap U_j$ for all i,j. We then can write $\|s\| = |s_i| \cdot e^{-\varphi_i}$. Note that since the s_i's satisfy a compatibility condition, we have something similar for the φ_i's: $\varphi_i - \varphi_j = \log|g_{ij}|$ on $U_i \cap U_j$. We will call the φ_i's the *(local) weights* of $\|\cdot\|$.

Example 5.2. A metric φ on $L = \mathcal{O}_X$ satisfies $\varphi_i = \varphi_j$ on $U_i \cap U_j$, which patch together to give a global function $\varphi\colon X \to \mathbf{R} \cup \{-\infty\}$. So, metrics on line bundles generalize real-valued functions on X.

5.1 Singular Metrics

Definition 5.3. We say a metric φ on L is *singular* if $\varphi_i \in L^1_{\mathsf{loc}}(U_i)$. In this case, we define a curvature form

$$dd^c\varphi := \frac{i}{\pi}\partial\overline{\partial}\varphi_i.$$

If the φ_i's are $\mathcal{C}^2$ or better, then you get an actual $(1,1)$-form with the appropriate regularity. If not, you have to take derivatives in the sense of distributions, that is, you get a $(1,1)$-form with coefficients being distributions.

Note 5.4.

- Check that this definition is independent of choices of U_i, so that φ is well-defined (you need to use that $\partial\overline{\partial}(\varphi_i - \varphi_j) = \partial\overline{\partial}(\log|g_{ij}|) = 0$).
- If $\varphi_i \in L^1_{\mathsf{loc}}(U_i)$, so that $dd^c\varphi$ is a closed $(1,1)$-current on X.
- The de Rham class of $dd^c\varphi$ is $c_1(L) \in H^2(X, \mathbf{C})$.

We write down some examples which will come up later.

Example 5.5 (Fubini–Study)**.** Let $X = \mathbf{P}^n$ with homogeneous coordinates $[z_0 : \cdots : z_n]$. Let $L = \mathcal{O}(1)$. Define $\|\cdot\|_{\mathrm{FS}}$ by declaring

$$\sum_{j=0}^{n} \|z_j\|^2_{\mathrm{FS}} = 1.$$

We want to show this uniquely characterizes $\|\cdot\|_{\mathrm{FS}}$. On $U_0 = \{z_0 \neq 0\} \cong \mathbf{C}^n$, we have local coordinates $(\xi_1, \dots, \xi_n)$. Then, we have

$$\|z_0\|_{\mathrm{FS}} = e^{-\varphi_0}, \qquad \|z_j\|_{\mathrm{FS}} = \|\xi_j \cdot z_0\|_{\mathrm{FS}} = |\xi_j| \cdot e^{-\varphi_0},$$

where φ_0 is the local weight of $\|\cdot\|_{\mathrm{FS}}$ on U_0. The condition above then becomes

$$1 = e^{-2\varphi_0}\left(1 + \sum_{j=1}^{n}|\xi_j|^2\right),$$

which implies

$$\varphi_0 = \frac{1}{2}\log\left(1 + \sum_{j=1}^{n}|\xi_j|^2\right).$$

Exercise 5.6.

1. $\|\cdot\|_{\mathrm{FS}}$ is smooth.
2. $dd^c\varphi_{\mathrm{FS}} > 0$ (see Definition 5.10).

Example 5.7. Let $s_1, \ldots, s_N$ be global sections of mL, and let (U_i) be a trivializing open cover of L, so that each s_j corresponds to a set of local sections $\{s_j^i \in \mathcal{O}_X(U_i)\}$. Define a metric φ on L by specifying the local weights as follows:

$$\varphi_i = \frac{1}{m} \log\biggl(\sum_{j=1}^{n} |s_j^i|^2\biggr).$$

Remark 5.8. The singular locus of φ in this example is $\bigcap_{j=0}^{n} s_j^{-1}(0)$, which corresponds to the base locus of the sublinear system generated by $s_1, \ldots, s_N$ in the complete linear system $|mL|$.

Example 5.9. Let $D = \sum \alpha_j D_j$ be a divisor for $\alpha_j \in \mathbf{Z}$, andlet $L = \mathcal{O}_X(D)$. Given a local section f, let $\|f\| = |f|$. To find the local weights, write $D_j \underset{\text{loc}}{\coloneqq} \{g_j = 0\}$. Then,

$$\|f\| = \biggl|f \cdot \prod_j g_j^{\alpha_j}\biggr| e^{-\sum \alpha_j \log|g_j|}.$$

The *Poincaré–LeLong equation* states that $dd^c\varphi = [D]$, where $[D]$ denotes the current of integration along D, and $\varphi \underset{\text{loc}}{=} \sum \alpha_j \log|g_j|$ is the additive version of the metric $\|\cdot\|$. Notice that $dd^c\varphi \geq 0$ (see Definition 5.10) if and only if D is effective.

Definition 5.10. A metric φ is *(semi)positive* if $dd^c\varphi > 0$ (resp. $dd^c\varphi \geq 0$), that is, $dd^c\varphi > \epsilon \cdot \omega$ (resp. $dd^c\varphi \geq \epsilon \cdot \omega$), where ω is the Kähler form (or any hermitian form) on X. L is *(semi)positive* if it admits a (semi)positive metric.

Example 5.11. If D is a divisor, then $L = \mathcal{O}_X(D)$ is semipositive when D is effective.

Fact 5.12. $dd^c\varphi \geq 0$ if and only if

$$\left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$$

is a semipositive definite matrix everywhere on our manifold. This is equivalent to saying the φ_i are plurisubharmonic; this is easy to see if the φ_i are $\mathcal{C}^2$.

Definition 5.13. Given a semipositive metric φ on L, we can define its multiplier ideal

$$\mathscr{J}(\varphi) \coloneqq \mathscr{J}(\varphi_i).$$

Exercise 5.14. Check this is well-defined.

Most results for algebraic multiplier ideals hold in this setting. For example,

Theorem 5.15 (Nadel)**.** *Let (X, ω) be a compact Kähler manifold, and let L be a holomorphic line bundle with singular metric ϕ such that $dd^c\phi \geq \epsilon \cdot \omega$ for some $\epsilon \in \mathcal{C}^0(X, \mathbf{R}_{>0})$, that is, some continuous function $\epsilon\colon X \to \mathbf{R}_{>0}$. Then, $H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathscr{J}(\phi)) = 0$ for all $i > 0$.*

We did not state the following theorem for algebraic multiplier ideals, but something similar holds in that setting. For analytic multiplier ideals, the statement is particularly nice:

Theorem 5.16 (Subadditivity)**.** *If φ, ϕ are singular semipositive metrics on line bundles, then*

$$\mathscr{J}(\varphi + \phi) \subset \mathscr{J}(\varphi)\mathscr{J}(\phi).$$

5.2 Algebraic and Analytic Positivity

Example 5.17. Let L be an ample line bundle on X. Then, there exists some m such that $L^{\otimes m}$ induces a closed embedding $X \overset{j}{\hookrightarrow} \mathbf{P}^N$ where $j^*\mathcal{O}(1) \cong L^{\otimes m}$. Thus, the metric $\varphi = \frac{1}{m} j^* \varphi_{\mathrm{FS}}$, where φ_{FS} is the Fubini–Study metric on $\mathcal{O}(1)$, is a positive smooth metric on L. This means that amplitude implies positivity. The converse is true by the Kodaira embedding theorem: if a compact complex manifold has a positive line bundle L, then L is ample and for some m, $L^{\otimes m}$ gives a closed embedding of the manifold into some $\mathbf{P}^N$.

Question 5.18. What concept from algebraic geometry does semipositivity correspond to?

Definition/Theorem 5.19 (Demailly, ...)**.** *L is pseudoeffective (psef) if and only if there exist a singular semipositive metric on L.*

Morally, this is shown in a similar way to the Kodaira embedding theorem: it boils down to an L^2 extension problem.

Examples 5.20.
- Effective divisors are psef;
- Ample line bundles are psef (also nef line bundles, or big line bundles).

Theorem 5.21 ("Uniform Global Generation," Siu 1998)**.** *If X is a smooth projective variety, then there exists an ample line bundle G such that for any psef line bundle (L, ϕ), then the sheaf $\mathcal{O}_X(K_X + G) \otimes \mathscr{J}(\phi)$ is globally generated.*

Example 5.22. The divisor $G = K_X + (\dim X + 1)A$ works, where A is any ample divisor, but this is hard to verify.

6 Zariski Decompositions

Lecture 5

Today we use analytic multiplier ideals to prove Fujita's approximation theorem, which concerns a version of the Zariski decomposition.

Let X be a smooth projective variety, and for the purposes of discussion, let L be an integral divisor over X. There are many definitions for what a Zariski decomposition should be. One definition is the following:

Definition 6.1. A *Zariski decomposition* of L consists a birational morphism $\mu \colon \tilde{X} \to X$ and a decomposition $\mu^* L = E + A$, where E is an effective $\mathbf{Q}$-divisor, and A is a nef $\mathbf{Q}$-divisor, satisfying the property that

$$H^0(X, mL) = H^0(X, \lfloor mA \rfloor) \quad \text{for all } m \geq 0.$$

This notion of Zariski decomposition is really nice because of the following:

Remark 6.2. If this cohomological condition holds, then

$$\operatorname{vol}(L) = \limsup_{m \to +\infty} \frac{h^0(X, mL)}{m^n/n!} = \operatorname{vol}(A) = (A^n). \tag{1}$$

Problem 6.3. Zariski decompositions as in Definition 6.1 don't exist in general.

We will instead construct an "approximate Zariski decomposition," for which the volume condition (1) does not hold, but we still get approximations for $\operatorname{vol}(L)$ in terms of $\operatorname{vol}(A)$ from above and below. This will be the statement of Fujita's approximation theorem.

6.1 Analytic Zariski Decompositions

Notation 6.4. We will refer to a metric on a line bundle as $h = \|\cdot\|$ (when we think of the metric multiplicatively), or $\phi = -\log\|\cdot\|$ (when we think of the metric additively).

The following is one of the main technical results we use in showing Fujita's approximation theorem.

Theorem 6.5. *If X is a compact complex manifold, and L is a psef line bundle on X, then there exists a unique (up to equivalence) metric $\phi = -\log h$ such that $dd^c\phi \geq 0$, and h has minimal singularities.*

We need to define some terms in this statement.

Note 6.6. We say that two metrics are *equivalent* if they are "infinite at the same places." We make more precise below:

1. $h_1 \sim h_2$ if there exists $C > 0$ such that $C^{-1}h_1 \leq h_2 \leq Ch_1$ (in terms of weight functions, this is saying that they are $-\infty$ at the same places), in which case we say h_1 and h_2 are *equivalent*. Note that this can be restated as saying there is an isometry between metrized line bundles (L, h_1) and (L, h_2).
2. $h_1 \leq h_2$ if there exists $C > 0$ such that $h_1 \leq C \cdot h_2$, in which case we say h_1 is *less singular* than h_2. h has *minimal singularities* if it is less singular than any other metric h'.

Proof Sketch. Fix a smooth reference metric h_0 with weight ϕ_0. Any singular metric on L can be written $h = h_0 e^{-\phi}$, and it is semipositive if $dd^c\phi \geq -dd^c\phi_0$. Now we can define a candidate for the minimal metric:

$$\phi_{\min} \coloneqq \sup\left\{\phi : dd^c\phi \geq -dd^c\phi_0 \text{ and } \sup_X \phi \leq 0\right\}.$$

The hard part is showing that these ϕ converge in the correct space of functions to give an actual singular semipositive metric $\phi_{\min}$ on L. □

We now define our notion of analytic Zariski decompositions.

Definition 6.7. Let X be a compact complex manifold, and let L be a line bundle on X. We say a singular semipositive metric h on L is an *analytic Zariski decomposition* if

$$H^0(X, mL) \cong H^0(X, mL \otimes \mathscr{J}(h^{\otimes m})) \text{ for all } m \geq 0.$$

Remark 6.8. This is strictly weaker than our Zariski decomposition from before.

The point of Theorem 6.5 is that it implies analytic Zariski decomposition exists, *assuming we have some semipositive metric on L to start with.*

Corollary 6.9. *If L is psef, then there exists an analytic Zariski decomposition.*

Proof. One inclusion of cohomology groups is clear: for any singular semipositive metric h,

$$H^0(X, mL) \supset H^0(X, mL \otimes \mathscr{J}(h^{\otimes m}))$$

by using the short exact sequence for the ideal sheaf $\mathscr{J}(h^{\otimes m})$.

Conversely, given $\sigma \in H^0(X, mL)$, define a metric h where

$$h(\xi) = \left|\frac{\xi}{\sigma(x)}\right|^{1/m}$$

for $\xi \in L_x$. Then, the local weight is $\frac{1}{m}\log|\sigma|$. Let $\phi_{\min}$ be the minimal singularity metric, i.e., the metric provided by Theorem 6.5. Then, we get $\frac{1}{m}\log|\sigma| \lesssim \phi_{\min}$ (locally), which is equivalent to saying $|\sigma|e^{-m\phi_{\min}}$ is locally bounded, and so $\sigma \in H^0(X, mL \otimes \mathscr{J}(m \cdot \phi_{\min})) = H^0(X, mL \otimes \mathscr{J}(h_{\min}^{\otimes m}))$. □

6.2 Fujita's Approximation Theorem

Theorem 6.5 and Corollary 6.9 are the key ingredients in showing that an "approximate Zariski decomposition" exists.

Theorem 6.10 (Fujita 1995)**.** *Let X be a projective variety of dimension n, and let L be a big line bundle on X. Then, for every $\epsilon > 0$, there exists a birational morphism $\mu\colon \tilde{X} \to X$ and a decomposition $\mu^*L = E + A$ where E is an effective $\mathbf{Q}$-divisor, A is an ample $\mathbf{Q}$-divisor, and*

$$\operatorname{vol} L < (A^n) + \epsilon.$$

Note that this morphism and the decomposition depend on ϵ.

Remark 6.11. Even though the statement of Theorem 6.10 only mentions an upper bound for $\mathrm{vol}(L)$, a lower bound is automatic as soon as we have an expression of the form $\mu^*L = E + A$. If this is the case, we can consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}}(kE) \longrightarrow \mathcal{O}_{kE}(kE) \longrightarrow 0,$$

where $k \in \mathbf{Z}$ is such that kE and kA are integral. Then, twisting by $\mathcal{O}_{\tilde{X}}(kA)$ and taking H^0 gives

$$H^0(\tilde{X}, kA) \hookrightarrow H^0(\tilde{X}, kA + kE) = H^0(X, kL)$$

and so $(A^n) = \mathrm{vol}(A) \le \mathrm{vol}(L)$.

We need two Lemmas to show this result. They take a while to prove, so we simply state them.

Lemma 6.12. *A line bundle L is big if and only if there exist $m_0 \in \mathbf{Z}_{>0}$ such that we can write $m_0 \cdot L = E + A$ where E is effective and A is ample.*

The direction $\Leftarrow$ is automatic, and for the direction $\Rightarrow$ you use a similar argument as in Remark 6.11.

Lemma 6.13. *Let L be big, and M any line bundle. Then, for all $\epsilon > 0$, there exists $m \in \mathbf{Z}_{>0}$ and a sequence $0 \le \ell_1 \le \ell_2 \le \cdots \nearrow +\infty$ such that*

$$h^0(X, \ell_j(mL - M)) \ge \frac{\ell_j^m m^n}{n!}(\mathrm{vol}(L) - \epsilon).$$

In particular, $\mathrm{vol}(mL - M) \ge m^n(\mathrm{vol}(L) - \epsilon)$ *for all* $m \gg 0$.

The broad strategy for showing Theorem 6.10 is as follows: we look at multiples mL of L, and to each of these we consider the multiplier ideal associated to the unique minimal singularity metric provided by Theorem 6.5. We then resolve this multiplier ideal sheaf, and when we lift everything upstairs, we get a Zariski decomposition basically automatically.

The issue with this strategy is that it doesn't quite work, since we need an "extra degree of freedom."

Proof of Fujita's approximation theorem. We first claim that it suffices to show that there exists a decomposition $\mu^*L = E + A$ for A a big and nef line bundle. We sketch the argument. Suppose there exists E an effective $\mathbf{Q}$-divisor and A a big and nef $\mathbf{Q}$-divisor, such that $\mu^*L = E + A$ and $(A^n) + \epsilon > \mathrm{vol}(L)$. By Lemma 6.12, we can write $A = E' + A'$ for E' effective and A' ample. Now note that

$$\mu^*L = E + A = (E + \epsilon E') + \underbrace{((1 - \epsilon)A + \epsilon A')}_{A''},$$

where A'' is ample for ϵ small enough since the ample cone is open. Then, note that

$$(A''^n) = (1 - \epsilon)^n (A^n) + \epsilon(\cdots),$$

which does not affect the inequality in the statement of Theorem 6.10, as long as ϵ is sufficiently small.

Now the "degree of freedom" we alluded to before is the line bundle G appearing in Siu's uniform global generation theorem (Theorem 5.21). Here, G has the property that for any psef line bundle (L, h), we have that $\mathcal{O}_X(G + L) \otimes \mathscr{J}(h)$ is globally generated.

Now we use Lemma 6.13 to say that

$$\mathrm{vol}(mL - G) \ge m^n(\mathrm{vol}(L) - \epsilon) \text{ for } m \gg 0.$$

In particular, for $m \gg 0$, we have that the right hand side is positive, and so $mL - G$ is psef, and we have a unique minimal singularity metric φ_m on $mL - G$ by Theorem 6.5. As φ_m is an analytic Zariski decomposition by Theorem 6.5, we have

$$H^0(X, \ell(mL - G)) = H^0(X, \ell(mL - G) \otimes \mathscr{J}(\ell \cdot \varphi_m))$$

for all $\ell \geq 0$. Note that using such a sequence φ_m of metrics is analogous to how in the algebraic setting, you can use asymptotic multiplier ideals to prove Fujita's approximation theorem.

Now resolve $\mathscr{J}(\varphi_m)$, that is, find a birational morphism $\mu\colon \tilde{X} \to X$ such that $\mu^*\mathscr{J}(\varphi_m) = \mathcal{O}_{\tilde{X}}(-E)$, where E is effective (note that Hironaka in fact implies more, but this property is all we need for the proof). Again by uniform global generation, we have that $\mu^*(\mathcal{O}_X(G + (mL - G)) \otimes \mathscr{J}(\varphi_m))$ is globally generated. But since

$$\mu^*(\mathcal{O}_X(G + (mL - G)) \otimes \mathscr{J}(\varphi_m)) = \mathcal{O}_{\tilde{X}}(m \cdot \mu^* L - E),$$

we have that $\mathcal{O}_{\tilde{X}}(m \cdot \mu^* L - E)$ is globally generated as well. This divisor $A = m \cdot \mu^* L - E$ is semiample and therefore nef for $m \gg 0$. This is exactly the decomposition we want, up to the factor m.

We now have candidates μ, E, A for the conclusion of Fujita's approximation theorem. It remains to show that they satisfy the condition on volumes. First, observe that

$$\mathcal{O}_{\tilde{X}}(-E)^{\otimes \ell} = \mu^*(\mathscr{J}(\varphi_m)^{\otimes \ell}) \supset \mu^*\mathscr{J}(\ell \cdot \varphi_m)$$

by the subadditivity theorem (Theorem 5.16). Next, observe that

$$\begin{aligned} H^0(\tilde{X}, \ell A) = H^0(\tilde{X}, \ell(m\mu^* L - E)) &\supset H^0(\tilde{X}, \mu^*\mathcal{O}_X(\ell m L) \otimes \mathscr{J}(\ell \cdot \varphi_m)) \\ &\supset H^0(\tilde{X}, \mu^*(\mathcal{O}_X(\ell(mL - G)) \otimes \mathscr{J}(\ell \cdot \varphi_m))) \end{aligned}$$

and that this last space is equal to $H^0(X, \mathcal{O}_X(\ell(mL-G)))$ by definition of the analytic Zariski decomposition. We are now almost done. By Lemma 6.13, we have

$$h^0(\tilde{X}, \ell A) \geq h^0(X, \ell(mL - G)) \geq \frac{\ell^n m^n}{n!}(\mathrm{vol}(L) - \epsilon),$$

and so $\mathrm{vol}(A) > m^n(\mathrm{vol}(L) - \epsilon)$. Now replacing E with $\frac{1}{m}E$ and A with $\frac{1}{m}A$, we are done. $\square$

We conclude with some remarks on our proof of Fujita's approximation theorem. Our proof is not Fujita's original proof from 1995; the argument is due to Demailly, Ein, and Lazarsfeld when they proved the subadditivity theorem (Theorem 5.16) in 2000. There is also an algebraic proof using asymptotic multiplier ideals $\mathscr{J}(\|mL\|)$ instead of the analytic multiplier ideals $\mathscr{J}(\varphi_m)$, which you can find in Lazarsfeld's book. The proof is morally the same.

This concludes the mini-course.