Stability

Ming Zhang[*]

August 22–27, 2016

Abstract

This mini-course will be an introduction to stability conditions on derived categories, wall-crossing and its applications to binational geometry of moduli spaces of sheaves and enumerative geometry on Calabi–Yau 3-folds (see details below). The schedule is as follows:

Lecture 1
- Motivation
- Basic notions: derived categories, triangulated categories, and Bridgeland's notion of stability conditions on triangulated categories.

Lecture 2
- Further discussion on stability conditions. I will give some examples of spaces of stability conditions.

Lectures 3–5
- Applications to the birational geometry of moduli spaces of sheaves: Every Bridgeland stability condition specifies a moduli space of Bridgeland-stable objects. I will explain the relation between wall-crossing for Bridgeland-stability and the minimal model program for the moduli space and discuss the result of Bayer–Macrí which shows that every minimal model of the moduli space of stable sheaves on a K3 surface appears as a moduli space of Bridgeland-stable objects on the K3 surface.
- Applications to enumerative geometry on Calabi–Yau threefolds: I will introduce Donaldson–Thomas (DT for short) invariants and Pandharipande–Thomas (PT for short) invariants of Calabi–Yau threefolds. I will explain Toda's proof of DT/PT correspondence by studying wall-crossing phenomena in the derived category.

I will focus on the second application and give more details.

Contents

---

[*]Notes were taken by Takumi Murayama, who is responsible for any and all errors.

1 History and Motivation

The goal of this course is to define stability conditions on triangulated categories, and describe applications of these stability conditions. These stability conditions, due to Bridgeland, have interesting applications in
- Birational geometry, e.g., that of $\overline{\mathcal{M}}_{g,n}$ and moduli spaces of stable sheaves on surfaces; and
- Enumerative geometry, where wall-crossing relates seemingly very different invariants.

The history of stability conditions is as follows. Stability conditions were first introduced classically on abelian categories. Abelian categories give rise to derived categories, which are examples of triangulated categories:

$$\text{abelian category} \implies \text{derived category} \subseteq \text{triangulated category}$$

We can then ask if there is some analogous notion of stability conditions on the triangulated side. These stability conditions on triangulated categories were introduced by T. Bridgeland, inspired by ideas of Douglas.

Douglas was a physicist, who was invited to speak at the 2002 ICM [Dou02]. His ideas were then formalized into something purely mathematical by Bridgeland in [Bri07]. The preprint first came out in 2002, and was not published until 2007, showing that in some sense his ideas were ahead of his time. The paper now has over 400 citations, showing how important his ideas have become.

We now elaborate on how stability conditions are applied to birational geometry. Let X be a scheme (or if you prefer, a complex manifold or variety). You can then consider the category $\mathsf{Coh}(X)$ of coherent sheaves on X. This is an abelian category, and has an associated derived category $\mathsf{D}(\mathsf{Coh}(X))$. To define a meaningful moduli space for objects in this category, we need to specify a subclass of objects, consisting of what we will call *stable* sheaves. This notion of stability depends on
- An ample divisor H on X; and
- Some cohomological data.

The easiest example of such a construction is the moduli space of stable curves over a curve C. The classical construction is for the *abelian* category $\mathsf{Coh}(X)$, but the version for the *derived* category has the advantage that changing the stability condition gives rise to interesting changes in the associated moduli spaces. These different spaces end up being connected by birational transformations, and so changing the stability condition gives rise to different birational models. In particular, for moduli spaces of stable sheaves on K3's, every birational model arises in this way.

2 Basic Notions and Examples

2.1 Abelian categories

We begin with the definition of an abelian category.

Definition 2.1. An abelian category $\mathcal{A}$ is a category satisfying the following properties:
1. For all objects $X, Y \in \mathsf{Obj}\,\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group;
2. $\mathcal{A}$ has a zero object, i.e., an object $0 \in \mathsf{Obj}\,\mathcal{A}$ such that $\mathrm{Hom}(0, X) = 0 = \mathrm{Hom}(X, 0)$;
3. Finite products $\prod G_i$ and coproducts $\bigoplus G_i$ exist and are isomorphic, where we recall that the universal property for the product of two objects is

   and the coproduct has the same universal property with arrows reversed;
4. Kernels and cokernels exist;
5. Monomorphisms and epimorphisms are normal, i.e., every monomorphism (resp. epimorphism) is the kernel (resp. cokernel) of a morphism, where we recall that a morphism $f\colon Y \to Z$ is a monomorphism if there exists a composition

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

   such that $f \circ g = 0$, then $g = 0$, and epimorphisms satisfy the same property with arrows reversed.

Examples 2.2. The following categories are abelian:

1. The category of right- (or left-) R-modules, for R a ring.
2. The category of (quasi-)coherent sheaves on a scheme X.
3. The category of representations of a quiver Q, which we define below.

Definition 2.3. A *quiver* $Q = (Q_0, Q_1)$ is a directed graph with vertices Q_0 and edges Q_1. A *representation* $V = (V_0, V_1)$ of Q is a set V_0 of vector spaces assigned to each vertex in Q_0, and a set V_1 of linear maps for each edge in Q_1.

Examples 2.4. We give some elementary examples of quivers and their representations.

1. Let V_0, V_1 be complex vector spaces, and let $\phi_i \in \mathrm{Hom}_{\mathbf{C}}(V_0, V_1)$. These can be assigned to the following quiver to give a representation:
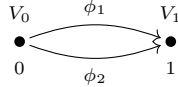
2. Quivers are allowed to have cycles, and so we can add two more arrows to our quiver and associate a representation as below:
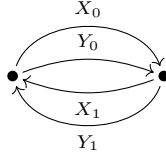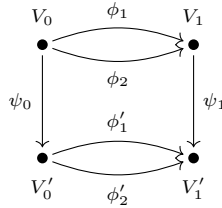

where we can impose conditions on these morphisms, e.g., $Y_1 X_0 = X_1 Y_0$ and $X_0 Y_1 = Y_0 X_1$.

3. Quivers are also allowed to have loops:

Definition 2.5. A *morphism of quiver representations* is a set of linear maps ψ_i between vector spaces assigned to vertices, commuting with the linear maps assigned to edges, e.g., maps ψ_i such that the diagram

commutes.

2.2 Stability conditions on abelian categories

We can now put a stability condition on an abelian category $\mathcal{A}$.

Example 2.6. Let C be a smooth complex projective curve (i.e., a Riemann surface), and consider the category $\mathsf{Coh}(C)$ of coherent sheaves on C. Let $E \in \mathsf{Coh}(C)$ be a vector bundle; E has a rank $\mathrm{rk}(E)$ and a degree $\deg(E)$. The *slope* of E is defined by

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} \quad (\mu(E) = +\infty \text{ if } E \text{ is torsion})$$

E is called *semistable* (resp. *stable*) if for all subsheaves $F \hookrightarrow E$ (resp. proper subsheaves), we have that $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$). Note that we can define the slope $\mu(F)$ for arbitrary coherent sheaves:

the rank is equal to the rank of F_η at the generic point of C, and the degree is given by forcing compatibility with Riemann–Roch:

$$\deg(F) = \chi(F) - \operatorname{rk} F \cdot (1 - g).$$

Recall that one of our motivations for stability conditions was to define moduli spaces of vector bundles on the curve C. A naïve strategy would be just to take *all* coherent sheaves on C, but this gives a moduli space that is not of finite type. Even if we wanted a moduli space for all rank 2 degree 0 vector bundles, it would still not be of finite type: for $C = \mathbf{P}^1$, all vector bundles $\mathcal{O}(-n) \oplus \mathcal{O}(n)$ for $n \geq 0$ appear in this family, but there is no finite type scheme that can parametrize all of these.

Luckily, this vector bundle is not semistable, and so restricting to the class of semistable sheaves does, in fact, give a good moduli space:

Theorem 2.7. *There exists a projective coarse moduli space $U_C(r, d)$ of semistable sheaves on C of rank r and degree d.*

The construction of this moduli space uses GIT.

Examples 2.8.
1. $U_C(1, 0) = \operatorname{Jac}(C)$;
2. If $C = \mathbf{P}^1$, then the moduli space of vector bundles with fixed first Chern class is a single point;
3. [Atiyah] If C is an elliptic curve, then $U_C(r, d) \cong \operatorname{Sym}^m(C)$, where $m = \gcd(r, d)$.

The slope function defined above satisfies the following key property:

Property 2.9 (See-saw). *Given a short exact sequence*

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

if $\mu(A) \leq \mu(E)$ (resp. $<, \geq, >$), then $\mu(E) \leq \mu(B)$ (resp. $<, \geq, >$).

The key idea behind the proof is what physicists call a "central charge" function on the Grothendieck group $K(\mathcal{A})$ of the abelian category $\mathcal{A}$. To define the central charge, we first define $K(\mathcal{A})$:

Definition 2.10. Let $\mathcal{A}$ be an abelian category. Then, the *Grothendieck group $K(\mathcal{A})$* of $\mathcal{A}$ is the abelian group generated by all isomorphism classes of objects in $\mathcal{A}$, modded out by relations of the form $[A] + [C] = [B]$ for every short exact sequence $0 \to A \to B \to C \to 0$.

We can now define what physicists call the "central charge":

Definition 2.11. The *central charge* of a coherent sheaf E is defined as

$$Z(E) = -\deg(E) + \operatorname{rk}(E)i.$$

Note that if $0 \to A \to E \to B \to 0$ is a short exact sequence, then
- $\operatorname{rk}(E) = \operatorname{rk}(A) + \operatorname{rk}(B)$;
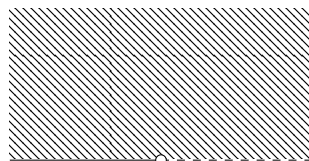- $\deg(E) = \deg(A) + \deg(B)$;
and so $Z(E) = Z(A) + Z(B)$. This means Z gives a homomorphism $Z \colon K(\mathcal{A}) \to \mathbf{C}$ from the Grothendieck group $K(\mathcal{A})$ to the complex numbers.

Proof. We first claim that the image of Z lies in the *semi-upper half plane*, that is, the region
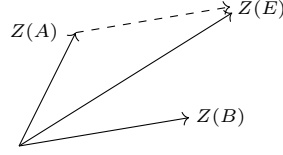
$$\mathbf{H} := \mathbf{R}_{<0} \cup \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\} \subseteq \mathbf{C},$$

which we can also picture visually as the region below:

If E is a torsion sheaf, then $\mathrm{rk}(E) = 0$ and $\deg(E) > 0$, since $\deg(E) = H^0(E)$. If E is a vector bundle, then $\mathrm{rk}(E) > 0$ and $\deg(E)$ can be arbitrary, so the image of Z is indeed contained in the semi-upper half plane.

Now if $\mu(A) \leq \mu(E)$, we get the following diagram of vectors:

where the slope of the vector $Z(A)$ is $-1/\mu(A)$, and that of $Z(B)$ is $-1/\mu(B)$. Since all three vectors lie in $\mathbf{H}$, we must have that if $\mu(A) \leq \mu(E)$, then $\mu(E) \leq \mu(B)$, and similarly for the other inequalities. $\quad\square$

Note that the proof only relied on the fact that the image of the central charge Z lied in $\mathbf{H}$.

One nice thing about semistable sheaves is that all coherent sheaves can be built up from semistable ones using the following filtration result:

Proposition 2.12 (Harder–Narasimhan filtration)**.** *For any coherent sheaf E on C, there exists a (unique) filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the successive quotients E_i/E_{i-1} are semistable of slope μ_i, where $\mu_1 > \mu_2 > \cdots > \mu_n$.

Example 2.13. On $\mathbf{P}^1$, a vector bundle E splits as $\bigoplus_{i=-\infty}^{\infty} \mathcal{O}(i)^{\oplus n_i}$, e.g., $\mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3)$. We then get a filtration

$$0 \longrightarrow \mathcal{O}(3) \longrightarrow \mathcal{O}(3) \oplus \mathcal{O}(2) \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3)$$

$$\begin{array}{ccc} \mathcal{O}(3) & \mathcal{O}(2) & \mathcal{O}(1) \\ \mu = 3 & \mu = 2 & \mu = 1 \end{array}$$

where we list the quotients E_i/E_{i-1} below the inclusion maps, together with their slopes.

Example 2.14. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbf{P}^1$ is an example of a decomposable bundle which is already semistable; in general, direct sums of isomorphic line bundles are semistable. Thus, the Harder–Narasimhan filtration for $\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)^{\oplus 2}$ is given by
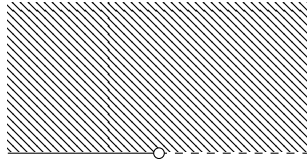
$$0 \longrightarrow \mathcal{O}(2)^{\oplus 2} \longrightarrow \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)^{\oplus 2}.$$

We now define stability for arbitrary abelian categories.

Definition 2.15. Let $\mathcal{A}$ be an abelian category and let $Z \colon K(\mathcal{A}) \to \mathbf{C}$ be a homomorphism such that for $0 \neq E \in \mathcal{A}$, we have

$$Z(E) \in \mathbf{H} = \{z = me^{i\pi\phi} \mid m > 0, \ \phi \in (0,1]\},$$

where $\mathbf{H}$ is the semi-upper half plane from before:

Then, we define the *phase*

$$\phi(E) = \frac{\arg(Z(E))}{\pi},$$

so that if $Z(E) = m(E)e^{\pi i \phi}$, then $\phi(E)$ is exactly this number ϕ. We say that $E \in \mathcal{A}$ is *semistable* (resp. *stable*) if for all subobjects $0 \to F \to E$ (resp. proper subobjects $0 \to F \to E$), we have $\phi(F) \leq \phi(E)$ (resp. $\phi(F) < \phi(E)$).

Even in this setting, we still have the analogue of the Harder–Narasimhan filtration:

Proposition 2.16 (Harder–Narasimhan filtration)**.** *For any object $E \in \mathcal{A}$, there exists a (unique) filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the successive quotients E_i/E_{i-1} are semistable of phase ϕ_i, where $\phi_1 > \phi_2 > \cdots > \phi_n$.

2.3 Derived categories

We now define derived categories $\mathsf{D}(\mathcal{A}), \mathsf{D}^+(\mathcal{A}), \mathsf{D}^-(\mathcal{A}), \mathsf{D}^b(\mathcal{A})$ associated to an abelian category $\mathcal{A}$.

Step 1. Define the category $\mathsf{Ch}(\mathcal{A})$ of chain complexes

$$\cdots \longrightarrow C_{-2} \longrightarrow C_{-1} \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots$$

where $C_i \in \mathcal{A}$ and the composition of two adjacent morphisms is 0. Morphisms in $\mathsf{Ch}(\mathcal{A})$ are given by commutative diagrams of the form

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{-2} & \longrightarrow & C_{-1} & \longrightarrow & C_0 & \longrightarrow & C_1 & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & B_{-2} & \longrightarrow & B_{-1} & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow \cdots
\end{array}
$$

Step 2. Define the homotopy category $\mathsf{K}(\mathcal{A})$ to be the category consisting of the same objects as $\mathsf{Ch}(\mathcal{A})$, but whose morphisms are the morphisms in $\mathsf{Ch}(\mathcal{A})$ quotiented out by the following equivalence relation. Consider the following diagram depicting two morphisms $f, g \colon C_\bullet \to B_\bullet$:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C_0 & \xrightarrow{i_0} & C_1 & \xrightarrow{i_1} & C_2 & \longrightarrow \cdots \\
& f_0 \Vert \downarrow g_0 & \overset{h_1}{\swarrow} & f_1 \Vert \downarrow g_1 & \overset{h_2}{\swarrow} & f_2 \Vert \downarrow g_2 & \\
\cdots \longrightarrow & B_0 & \xrightarrow{j_0} & B_1 & \xrightarrow{j_1} & B_2 & \longrightarrow \cdots
\end{array}
$$

f and g are called *homotopy equivalent* with *homotopy* $h = (h_i)$ if there exists a sequence of morphisms $h_i \colon C_i \to B_{i-1}$ such that

$$f_i - g_i = j_{i-1} h_i + h_{i+1} i_i$$

for all i. Note that this implies $f_* = g_*$ as maps $H^i(C_\bullet) \to H^i(B_\bullet)$.

Step 3. The derived category $\mathsf{D}(\mathcal{A})$ has the same objects as $\mathsf{K}(\mathcal{A})$ or $\mathsf{Ch}(\mathcal{A})$, but morphisms $X \xrightarrow{f} Y$ are given by roofs

$$
\begin{array}{ccc}
& X' & \\
{\scriptstyle h}\swarrow_{\text{qis}} & & \searrow{\scriptstyle g} \\
X & & Y
\end{array}
$$

where g, h are morphisms in $\mathsf{K}(\mathcal{A})$, and "qis" denotes a quasi-isomorphism, that is, a morphism of chain complexes that induces isomorphisms on cohomology. We compose morphisms by constructing a diagram

$$
\begin{array}{ccccc}
& & X'' & & \\
& \swarrow_{\text{qis}} & & \searrow & \\
& X' & & Y' & \\
{\scriptstyle h}\swarrow_{\text{qis}} & & \searrow{\scriptstyle g} \quad \swarrow_{\text{qis}} & & \searrow \\
X & & Y & & Z
\end{array}
$$

Note that you need $\mathsf{K}(\mathcal{A})$ because otherwise X'' may not exist such that $X'' \to X'$ is a quasi-isomorphism. See [GM03, Thm. III.4.4].

In this way, all quasi-isomorphisms have been "inverted" in $\mathsf{D}(\mathcal{A})$.

Definition 2.17. We obtain the other variants of the derived category by starting with different subcategories of $\mathsf{Ch}(\mathcal{A})$ with different boundedness properties.

$\mathsf{D}^+(\mathcal{A})$ starts with $\mathsf{Ch}^+(\mathcal{A})$, the chain complex of bounded below chain complexes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow C_{-1} \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots$$

$\mathsf{D}^-(\mathcal{A})$ starts with $\mathsf{Ch}^-(\mathcal{A})$, the chain complex of bounded above chain complexes

$$\cdots \longrightarrow C_{-1} \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$\mathsf{D}^b(\mathcal{A})$ starts with $\mathsf{Ch}^b(\mathcal{A})$, the chain complex of bounded chain complexes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow C_{-1} \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Note that we have cohomology functors $H^n \colon \mathsf{Ch}(\mathcal{A}) \to \mathcal{A}$ defined by $H^i(E_\bullet) = \ker d_i / \operatorname{im} d_{i-1}$. These descend to functors on $\mathsf{K}(\mathcal{A})$ and $\mathsf{D}(\mathcal{A})$:


2.4 Stability conditions for quiver representations

We now define more examples of stability conditions. Consider an abelian category $\mathcal{A}$ such that the Grothendieck group $K(\mathcal{A})$ is freely generated by some isomorphism classes $E_1, \ldots, E_n$. Then, choosing arbitrary $z_1, \ldots, z_n \in \mathbf{H}$, we can define a central charge $Z \colon K(\mathcal{A}) \to \mathbf{H}$ by having $Z(E_i) = z_i$, and thereby define a stability condition on $\mathcal{A}$. This is how we will define stability conditions for quiver representations.

Fix a quiver Q with vertices Q_0 and edges Q_1. Recall that a quiver representation $V = (V_0, V_1)$ is an assignment of vector spaces in a set V_0 to each vertex, and linear maps in a set V_1 to each edge; for example
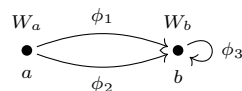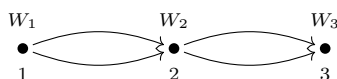

Quiver representations of a quiver Q form an abelian category Rep_Q. Assuming that Q has no loops or oriented cycles, the Grothendieck group is freely generated as
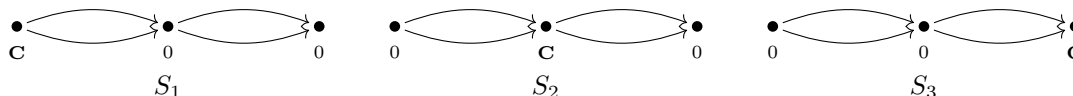
$$K(\mathsf{Rep}_Q) = \langle S_i \rangle_{1 \leq i \leq |Q_0|},$$

where the S_i are defined by having one copy of $\mathbf{C}$ assigned to the ith vertex, and having 0 assigned to every other vertex.

Example 2.18. Consider the following quiver:
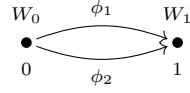

The quiver representations S_i are the following:

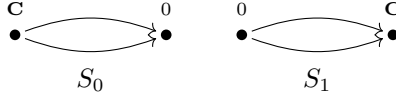Then, we choose $z_1, z_2, z_3 \in \mathbf{H}$, and define $Z(S_i) = z_i$, so that

$$Z \colon K(\mathsf{Rep}_Q) \longrightarrow \mathbf{C}$$
$$V \longmapsto \dim(W_1)z_1 + \dim(W_2)z_2 + \dim(W_3)z_3.$$

where W_i is the vector space assigned to the ith vertex.

7

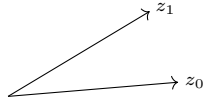**Example 2.19** (Kronecker quiver)**.** Consider the *Kronecker quiver*

for which the quiver representations S_i are the following:

As before, we can choose $z_0, z_1 \in \mathbf{H}$ to give a stability condition on Rep_Q, and ask what the stable representations are. The answer depends on the choice of z_0, z_1:

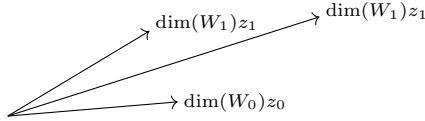**Case 1.** The phase of z_1 is greater than that of z_0:

Let V be a representation, in which case $Z(V) = \dim(W_0)z_0 + \dim(W_1)z_1$. Then, there are three subcases:

    1. Suppose $\dim W_0 \geq 1$ and $\dim W_1 \geq 1$. In this case, the subrepresentation V' given by
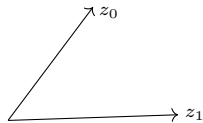

has $Z(V') = \dim(W_1)z_1$, which has a larger phase:

    Thus, V' destabilizes V, and so V is not stable.

    2. Suppose $\dim W_0 \geq 1$ and $\dim W_1 = 0$. Then, S_0 is a destabilizing subrepresentation unless $\dim W_0 = 1$.

    3. Suppose $\dim W_0 = 0$ and $\dim W_1 \geq 1$. Then, S_1 is a destabilizing subrepresentation unless $\dim W_1 = 1$.

We therefore see that the only stable representations are S_0 and S_1.

Case 2. The phase of z_0 is greater than that of z_1:

We will classify stable representations with dimension vector $(1, 1)$, that is, representations V of the form
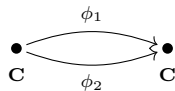
Then, V is stable if and only if S_0 does not exist as a subrepresentation. This is equivalent to saying $(\phi_1, \phi_2) \neq (0, 0)$, and so since for any scalar $\lambda \in \mathbf{C}^*$, the maps (ϕ_1, ϕ_2) and $(\lambda\phi_1, \lambda\phi_2)$ give rise to isomorphic quiver representations, we see that isomorphism classes of stable representations with dimension vector $(1, 1)$ are parametrized by

$$\left\{(\phi_1, \phi_2) \in \mathbf{C}^2 \setminus (0, 0)\right\}\Big/\mathbf{C}^* \cong \mathbf{P}^1.$$

This gives an example of wall-crossing: changing the stability condition changes the moduli space of stable representations with dimension vector $(1,1)$ from the empty set in Case 1, to $\mathbf{P}^1$ in Case 2.

Example 2.20 (P_{n+1})**.** We can generalize the previous example as follows. Consider the quiver P_{n+1}:
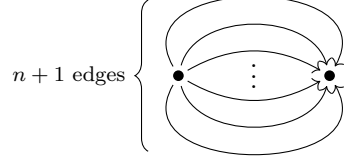
If the phase of z_0 is greater than that of z_1, then the same analysis as in Case 2 of Example 2.19 shows that isomorphism classes of stable representations with dimension vector $(1,1)$ are parametrized by $\mathbf{P}^n$.

For quivers with cycles, you can still define a central charge by

$$Z(V) = \sum_{i=0}^{r} \dim(W_i) z_i$$

for, e.g., a quiver

and still get a group homomorphism $K(\mathsf{Rep}_{Q,R}) \to \mathbf{C}$.
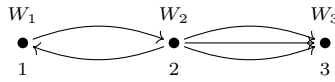
2.5 Triangulated categories

The derived category $\mathsf{D}(\mathcal{A})$ is an example of a triangulated category, which we will now define.

Definition 2.21. A category $\mathcal{D}$ is a *triangulated category* if it is an additive category (i.e., all Hom sets are abelian groups) together with
1. an autoequivalence (i.e., an equivalence of categories $\mathcal{D} \to \mathcal{D}$) denoted by $X \mapsto X[1]$ called the *shift* or *translation* functor; and
2. a set of *distinguished triangles*
$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

satisfying the following axioms:
   (a) $X \xrightarrow{\text{id}} X \to 0 \to X[1]$ is distinguished;
   (b) For any morphism $u \colon X \to Y$, there exists a third object Z called the *mapping cone* of u such that $X \xrightarrow{u} Y \to Z \to X[1]$ is a triangle (the mapping cone is unique, but only up to non-unique isomorphism);
   (c) Any triangle isomorphic to a distinguished triangle is distinguished, where an isomorphism is a commutative diagram
$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X[1]
\end{array}$$

with the vertical morphisms being isomorphisms;
   (d) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished, then

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is also distinguished;

9

(e) If we have a diagram

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}
$$

it can be completed with a (non-unique) map $Z \to Z'$;

(f) [Octahedral axiom] Given morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, then we can complete the commutative triangle below with distinguished triangles $A \to B \to D \to A[1]$, $A \to C \to E \to A[1]$, and $B \to C \to F \to B[1]$, so that $D \to E \to F \to D[1]$ is also distinguished, such that the diagram below commutes:

where squiggly arrows in a triple $X \to Y \to Z \rightsquigarrow$ denotes that the triple forms a distinguished triangle $X \to Y \to Z \to X[1]$.

We think of the octahedral axiom as the triangulated version of the fact that $(C/A)/(B/A) \cong C/B$ in an abelian category.

To motivate the notion of a t-structure, we ask the following:

Question 2.22. Consider the category $\mathsf{D}^b(\mathcal{A})$. Note that $\mathcal{A}$ is a subcategory of $\mathsf{D}^b(\mathcal{A})$, by identifying $\mathcal{A}$ with the subcategory of objects with only the degree 0 slot being nonzero. But given $\mathsf{D}^b(\mathcal{A})$, how can we recover the category $\mathcal{A}$?

It turns out that we cannot recover $\mathcal{A}$ from $\mathsf{D}^b(\mathcal{A})$ without a t-structure, a notion invented by Beĭlinson–Bernstein–Deligne in [BBD82].

Definition 2.23. Let $\mathcal{D}$ be a triangulated category. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a pair of saturated (i.e., closed under isomorphism) full subcategories of $\mathcal{D}$. We will denote $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ and $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$. Then, we say $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure if the pair satisfies the following properties:
1. If $X \in \mathcal{D}^{\leq 0}$, and $Y \in \mathcal{D}^{\geq 1}$, then $\mathrm{Hom}(X, Y) = 0$;
2. $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$.
3. For every $X \in \mathcal{D}$, there exists a distinguished triangle

$$
\begin{array}{ccccccc}
A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & A[1] \\
\cap & & & & \cap & & \\
\mathcal{D}^{\leq 0} & & & & \mathcal{D}^{\geq 1} & &
\end{array}
$$

This assignment is functorial; we denote $\tau_{\leq 0} X$ to be the object A above, and $\tau_{\geq 1}$ to be the object B above. $\tau_{\leq 0}$ and $\tau_{\geq 1}$ are called *truncation functors*.

Example 2.24. For $\mathcal{D} = \mathsf{D}(\mathcal{A})$, recall that objects are chain complexes

$$
E_\bullet := \{E_{-1} \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots\}
$$
$$
E_\bullet[1] := \{E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots\}
$$

and the shift $[1]$ shifts everything to the left as shown above. We then have a natural t-structure coming from cohomology:

$$\mathcal{D}^{\leq 0} = \{X \in \mathsf{D}(\mathcal{A}) \mid H^i(X) = 0 \; \forall i > 0\}$$
$$\mathcal{D}^{\geq 0} = \{X \in \mathsf{D}(\mathcal{A}) \mid H^i(X) = 0 \; \forall i < 0\}$$

The truncation functors are given by

$$\tau_{\leq 0} \colon \mathcal{D} \longrightarrow \mathcal{D}^{\leq 0}$$

$$\{X^{-2} \to X^{-1} \to X^0 \overset{d}{\to} X^1 \to X^2 \to \cdots\} \longmapsto \{X^{-2} \to X^{-1} \to X^0 \to \ker d \to 0 \to 0 \to \cdots\}$$

$$\tau_{\geq 1} \colon \mathcal{D} \longrightarrow \mathcal{D}^{\geq 1}$$

$$\{X^{-2} \to X^{-1} \to X^0 \overset{d}{\to} X^1 \to X^2 \to \cdots\} \longmapsto \{0 \to 0 \to X^0/\ker d \to X^1 \to X^2 \to \cdots\}$$

and this gives a distinguished triangle

$$\tau_{\leq 0} X \longrightarrow X \longrightarrow \tau_{\geq 1} X \longrightarrow \tau_{\leq 0} X[1]$$

Definition 2.25. $\mathcal{A}^{\#} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the *heart* or *core* of the t-structure.

Proposition 2.26. $\mathcal{A}^{\#}$ *is abelian.*

Example 2.27. If $\mathcal{D} = \mathsf{D}(\mathcal{A})$, and you choose the standard t-structure, then you can show that $\mathcal{A}^{\#} = \mathcal{A}$. However, it is not true in general that for an arbitrary triangulated category $\mathcal{D}$, we have $\mathsf{D}(\mathcal{A}^{\#}) \cong \mathcal{D}$, e.g., by taking the t-structure $(\mathcal{D}, 0)$ on $\mathcal{D}$.

We can give an equivalent formulation of a t-structure that is bounded, that is

Definition 2.28. A t-structure on a triangulated category $\mathcal{D}$ is *bounded* if

$$\bigcap_n \mathsf{Obj}\, \mathcal{D}^{\leq n} = \bigcap_n \mathsf{Obj}\, \mathcal{D}^{\geq n} = \{0\}.$$

If our t-structure is bounded, then we can reformulate the information contained in the t-structure in terms of the heart:

Definition 2.29. A *heart* of a bounded t-structure is a full additive subcategory $\mathcal{A}^{\#} \subset \mathcal{D}$ such that
1. If $k_1 > k_2$, then $\mathrm{Hom}(\mathcal{A}^{\#}[k_1], \mathcal{A}^{\#}[k_2]) = 0$;
2. For every nonzero object E in $\mathcal{D}$, there exist integers $k_1 > k_2 > \cdots > k_n$ and a sequence of exact triangles

$$0 = E_0 \longrightarrow E^1 \longrightarrow E^2 \to \cdots \to E^{n-1} \longrightarrow E^n = E$$
$$A_1 \qquad\qquad A_2 \qquad\qquad\qquad A_n$$

where $A_i \in \mathcal{A}^{\#}[k_i]$ for all i.

2.6 Tilting

Definition 2.30. Let $\mathcal{A}$ be an abelian category. A *torsion pair* $(\mathcal{T}, \mathcal{F})$ is a pair of full additive subcategories of $\mathcal{A}$ such that
1. $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$;
2. For all $E \in \mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

$\mathcal{T}$ is the "torsion part" and $\mathcal{F}$ is the "free part."

Examples 2.31.
1. Let $\mathcal{A} = \mathsf{Coh}(X)$. Then, the pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T}$ is the category of torsion sheaves, and $\mathcal{F}$ is the category of torsion-free sheaves, is a torsion pair.
2. Let $\mathcal{A} = \mathsf{Coh}(C)$, for C a curve, and let $\mu \in \mathbf{R}$. Then, let $\mathcal{A}_{\geq \mu}$ (resp. $\mathcal{A}_{<\mu}$) be the subcategory consisting of all objects E such that every factor in the Harder–Narasimhan filtration for E has slope $\geq \mu$ (resp. $< \mu$). Recall that the Harder–Narasimhan filtration is a filtration of the form

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

   and letting $\mu_i = \mu(E_i/E_{i-1})$, each factor E_i/E_{i-1} is semistable with respect to μ with slope μ_i, and $\mu_1 > \mu_2 > \cdots > \mu_n$. An arbitrary object E then fits in a short exact sequence
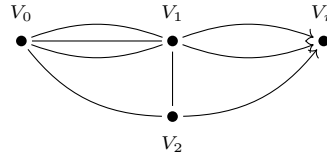
$$0 \longrightarrow E_i \longrightarrow E \longrightarrow E/E_i \longrightarrow 0$$

   where i is the largest number such that $\mu_i \geq \mu$. This shows that $(\mathcal{A}_{\geq \mu}, \mathcal{A}_{<\mu})$ is a torsion pair. We mention the following:

   Fact 2.32. $\mathrm{Hom}(E, F) = 0$ if $\mu_-(E) > \mu_+(F)$, where μ_- (resp. μ_+) are the smallest (resp. largest) slopes appearing in the Harder–Narasimhan filtration.

   This motivates property (1) of t-structures in Definition 2.23.
3. Consider a quiver such that a vertex n is a sink, e.g.

   Let $\mathcal{A} = \mathsf{Rep}_Q$, let $\mathcal{T}$ be the subcategory of representations concentrated at vertex n, and let $\mathcal{F}$ be the subcategory of representations with $V_n = 0$. Then, $(\mathcal{T}, \mathcal{F})$ is a torsion pair, since the subobject
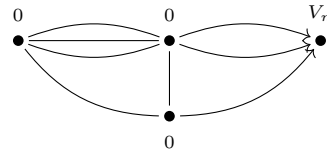
   is in $\mathcal{T}$ with quotient in $\mathcal{F}$.
   In general, even if n is not a sink, we can still let $\mathcal{T}$ be the same subcategory as before consisting of representations $S_n^{\oplus k}$, where S_n is the simple one-dimensional representation concentrated at n, and let $\mathcal{F}$ be the subcategory of representations V such that $\mathrm{Hom}(S_n, V) = 0$. More explicitly, these are representations V satisfying

$$\bigcap_{\substack{j \text{ such that } e_j \\ \text{goes out of} \\ \text{vertex } n}} \ker\!\left(V_n \xrightarrow{\phi_j} V_{\phi_j}\right) = 0.$$

Proposition 2.33 (Tilting). *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$, we can define a new heart for $\mathsf{D}^b(\mathcal{A})$*

$$\mathcal{A}^{\#} = \{E \in \mathsf{D}^b(\mathcal{A}) \mid H^0(E) \in \mathcal{T}, \ H^{-1}(E) \in \mathcal{F}, \ H^i(E) = 0 \text{ for } i \neq 0, -1\},$$

where $H^{-1}(X) := H^0(X[-1])$.

It is also possible to define tilting for an arbitrary triangulated category $\mathcal{D}$ with a t-structure with heart $\mathcal{A}^{\#}$, using truncation functors instead of cohomology functors.

Remark 2.34. An object $E \in \mathsf{Obj}\, \mathcal{A}^{\#}$ is a complex

$$0 \longrightarrow F \overset{u}{\longrightarrow} G \longrightarrow 0$$

where $\ker u \in \mathcal{F}$ and $\operatorname{cok} u \in \mathcal{T}$, in which case you get an exact sequence

$$0 \longrightarrow \ker u \longrightarrow F \overset{u}{\longrightarrow} G \longrightarrow \operatorname{cok} u \longrightarrow 0,$$

i.e., $E \in \operatorname{Ext}^2(\mathcal{T}, \mathcal{F}) = \operatorname{Ext}^1(\mathcal{T}, \mathcal{F}[1])$. Here we use that if F, G are objects of $\mathcal{A}$, then

$$\operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(F[p], G[q]) = \begin{cases} 0 & p > q \\ \operatorname{Ext}^{q-p}(F, G) & p \leq q \end{cases}$$

We can visualize tilting as follows:

$\mathcal{A}[1]$			$\mathcal{A}[0]$			$\mathcal{A}[-1]$	
$\mathcal{T}[1]$	$\mathcal{F}[1]$	$\mathcal{T}[0]$	$\mathcal{F}[0]$	$\mathcal{T}[-1]$	$\mathcal{F}[-1]$
	$\mathcal{A}^{\#}[0]$			$\mathcal{A}^{\#}[-1]$		

Example 2.35. Let $X = \mathbf{P}^1$, and let $\mathcal{A} = \mathsf{Coh}(X)$. Let $(\mathcal{T}, \mathcal{F}) = (\mathcal{A}_{\geq 0}, \mathcal{A}_{<0})$ to obtain the tilted heart $\mathcal{A}^{\#}$. We can also consider Rep_Q for the Kronecker quiver

Note that $\mathsf{Coh}(\mathbf{P}^1)$ and Rep_Q are very different abelian categories, since e.g., the latter category has Jordan–Hölder filtrations. However, their derived categories are equivalent: there exists an equivalence of categories

$$\Phi_T \colon \mathsf{D}^b(\mathsf{Coh}(\mathbf{P}^1)) \overset{\sim}{\longrightarrow} \mathsf{D}^b(\mathsf{Rep}_Q),$$

defined by the *tilting sheaf* (in this case a bundle) $T = \mathcal{O} \oplus \mathcal{O}(1)$, which we will define below. Under this equivalence, we have $\Phi_T(\mathcal{A}^{\#}) = \mathsf{Rep}_Q$. This shows both $\mathsf{Coh}(X)$ and Rep_Q exist as hearts of different t-structures on $\mathcal{D}$.

Definition 2.36. Let X be a smooth projective variety, for simplicity over $\mathbf{C}$ (this should work for any algebraically closed field, maybe assuming the characteristic is zero). Then, a coherent sheaf T on X is a *tilting sheaf* if

1. The *tilting algebra* $A = \operatorname{End}_{\mathcal{O}_X}(T)$ has finite global dimension, that is

$$\sup \{\text{projective dimension of left } A\text{-module}\} < \infty;$$

2. $\operatorname{Ext}^k_{\mathcal{O}_X}(T, T) = 0$ for all $k > 0$;
3. T *classically generates* the derived category $\mathsf{D}^b(\mathsf{Coh}\, X)$, that is, $\mathsf{D}^b(\mathsf{Coh}\, X)$ can be obtained from T by repeated iterations of taking cones, direct summands, shifts, etc..

Theorem 2.37 (Baer, Bondal)**.** *Let T be a tilting sheaf on X, and let $A = \operatorname{End}_{\mathcal{O}_X}(T)$ be the tilting algebra. Then, the functors*

$$\begin{aligned} F(-) &:= \operatorname{Hom}_{\mathcal{O}_X}(T, -) \colon && \mathsf{Coh}(X) \longrightarrow \mathsf{mod}(A^{\mathsf{op}}) \\ G(-) &:= - \otimes_A T \colon && \mathsf{mod}(A^{\mathsf{op}}) \longrightarrow \mathsf{Coh}(X) \end{aligned}$$

induce equivalences of tirangulated categories

$$\begin{aligned} \mathbf{R}F \colon && \mathsf{D}^b(\mathsf{Coh}(X)) &\longrightarrow \mathsf{D}^b(\mathsf{mod}(A^{\mathsf{op}})) \\ \mathbf{R}G \colon && \mathsf{D}^b(\mathsf{mod}(A^{\mathsf{op}})) &\longrightarrow \mathsf{D}^b(\mathsf{Coh}(X)) \end{aligned}$$

Example 2.38. In the previous example, the derived functor is

$$\mathsf{D}^b(\mathsf{Coh}(X)) \longrightarrow \mathsf{D}^b(\mathsf{mod}(\operatorname{End}(\mathcal{O} \oplus \mathcal{O}(1)))) = \mathsf{D}^b(\mathsf{mod}(\mathbf{C}Q)) = \mathsf{D}^b(\mathsf{Rep}_Q),$$

where elements of $\mathsf{mod}(\mathbf{C}Q)$ can be thought of as paths in Q.

You can compute examples with more complicated varieties X and associated quivers Q; see [Cra07].

2.7   Stability on triangulated categories

Definition 2.39. A *slicing* $\mathcal{P}$ of $\mathcal{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi)$ for $\phi \in \mathbf{R}$ satisfying
1. $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
2. If $\phi_1 > \phi_2$, then $\operatorname{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$;
3. If E is a nonzero object in $\mathcal{D}$, then there is a sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ and a sequence of distinguished triangles

$$0 = E_0 \longrightarrow E^1 \longrightarrow E^2 \to \cdots \to E^{n-1} \longrightarrow E^n = E$$
$$A_1 \qquad A_2 \qquad\qquad A_n$$

such that $A_i \in \mathcal{P}(\phi_i)$ for all i.

Note this is a refinement of the definition of a heart, since we allow $\mathcal{P}(\phi_i)$ to be indexed by real numbers, not just integers.

Remarks 2.40.
1. Objects in $\mathcal{P}(\phi)$ are called *semistable of phase ϕ*.
2. Define $\phi_\mathcal{P}^+(E) = \phi_1$ and $\phi_\mathcal{P}^-(E) = \phi_n$. One can show that for $A, B \in \mathcal{D}$, if $\phi_\mathcal{P}^-(A) > \phi_\mathcal{P}^+(B)$, then $\operatorname{Hom}(A, B) = 0$. This generalizes the notion of slope from before.
3. If $\mathcal{P}(\phi) = \emptyset$ for $\phi \notin \mathbf{Z}$, then slicing is the same as the notion of a bounded t-structure with heart $\mathcal{P}(0)$. Otherwise, we define the heart as

$$\mathcal{A} = \mathcal{P}((0,1]) = \{E \mid \phi^+(E) \le 1, \ \phi^-(E) > 0\}.$$

We can now almost define stability conditions for triangulated categories. We first need to define the analogue of the Grothendieck group $K(\mathcal{A})$ for triangulated categories:

Definition 2.41. Let $\mathcal{D}$ be a triangulated category. Then, the *Grothendieck group* $K(\mathcal{D})$ of $\mathcal{D}$ is the abelian group generated by all isomorphism classes of objects in $\mathcal{D}$, modded out by relations of the form $[A] + [C] = [B]$ for every distinguished triangle $A \to B \to C \to A[1]$.

Remark 2.42. If $\mathcal{A}$ is an abelian category, then $K(\mathcal{A}) \cong K(\mathsf{D}(\mathcal{A}))$.

Definition 2.43. A *stability condition on $\mathcal{D}$* is a pair $(Z, \mathcal{P})$ where $Z \colon K(\mathcal{D}) \to \mathbf{C}$ is a group homomorphism, and $\mathcal{P}$ is a slicing, such that for all $0 \ne E \in \mathcal{P}(\phi)$ where $\phi \in \mathbf{R}$, we have $Z(E) = m(E) \cdot e^{i\pi\phi}$ for $m(E) > 0$.

Theorem 2.44 (Bridgeland)**.** *The following data are equivalent:*

$$\left\{ \begin{array}{c} \textit{stability conditions} \\ \sigma = (Z, \mathcal{P}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{bounded t-structures with hearts } \mathcal{A}, \textit{ together with a stability} \\ \textit{condition } Z_\mathcal{A} \textit{ on } \mathcal{A}, \textit{ satisfying the Harder–Narasimhan property} \end{array} \right\}$$

Proof. Given $(\mathcal{A}, Z_\mathcal{A})$, we define a slicing as follows: for $0 < \phi \le 1$, define

$$\mathcal{P}(\phi) = \left\{ \begin{array}{c} \text{semistable objects with phase } \phi \\ \text{with respect to } Z_\mathcal{A} \end{array} \right\} \subset \mathcal{A}$$

where the phase is that associated to the central charge $Z(\mathcal{A}) \colon K(\mathcal{A}) \to \mathbf{H}$. For $n \in \mathbf{N}$ and $0 < \phi \le 1$, we define $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n]$. You then check that this slicing satisfies the definition of a slicing. In particular, for the third property, suppose E is given. Then, we have a filtration

$$0 = \tilde{E}_0 \longrightarrow \tilde{E}_1 \longrightarrow \tilde{E}_2 \to \cdots \to \tilde{E}_{n-1} \longrightarrow \tilde{E}_n = E$$
$$\tilde{A}_1 \qquad \tilde{A}_2 \qquad\qquad \tilde{A}_n$$

where $\tilde{A}_i \in \mathcal{A}[k_i]$ for all i, and $k_1 > \cdots > k_n$. Then, we get filtrations for $\tilde{A}_i$ in $\mathcal{A}$ by semistable objects with respect to $Z_{\mathcal{A}}$.

In the other direction, if you are given $\sigma = (Z, \mathcal{P})$, we define $\mathcal{A}$ to be the smallest extension closed subcategory generated by $\mathcal{P}(\phi)$ where $0 < \phi \leq 1$, in which case $Z_{\mathcal{A}}$ is just the restriction of Z to $\mathcal{A}$, i.e., we say $E \in \mathcal{A}$ is semistable with respect to $Z_{\mathcal{A}}$ of phase ϕ if $E \in \mathcal{P}(\phi)$. $\qquad\square$

Example 2.45. Let $\mathcal{D} = \mathsf{D}^b(\mathsf{Coh}(C))$ for C a curve, with heart $\mathcal{A} = \mathsf{Coh}(C)$. We had the central charge $Z_{\mathcal{A}} = -\deg + i \cdot \mathrm{rk}$. Semistable objects in $\mathcal{D}$ then consist of shifts of sheaves of semistable coherent sheaves and 0-dimensional torsion sheaves.

Remark 2.46. Existence of stability conditions on $\mathsf{D}^b(\mathsf{Coh}(X))$ where X is a smooth projective Calabi–Yau threefold is unknown. You can instead study when X is a K3 surface, or when X is an *open* Calabi–Yau threefold.

We will now discuss advantages to having the derived side of the story in Bridgeland's theorem. The reason is that we have "more freedom." The key is that stability conditions $\sigma = (Z, \mathcal{P})$ form a manifold called the *stability manifold*.

Example 2.47. The stability manifold for $\mathbf{P}^1$ is $\mathbf{C}^2$.

Definition 2.48 ([KS08])**.**

1. Fix a finite dimensional lattice Λ and assume that there exists a map $\lambda\colon K(\mathcal{D}) \to \Lambda$, such that the stability function Z factors via Λ:

$$\lambda\colon K(\mathcal{D}) \longrightarrow \Lambda$$
$$Z \searrow \qquad \swarrow$$
$$\mathbf{C}$$

2. [Support property] Let $\|\cdot\|$ be an arbitrary norm on $\Lambda_{\mathbf{R}} := \Lambda \otimes \mathbf{R}$. Assume σ satisfies a "support property":

$$\inf \left\{ \frac{|Z(E)|}{\|E\|} \;\middle|\; E \text{ is } \sigma\text{-semistable} \right\} > 0$$

3. Let $\sigma^1 = (\mathcal{P}, Z^1)$, $\sigma^2 = (\mathcal{Q}, Z^2)$ be two stability conditions. We define a metric on slicings

$$d_S(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^+_{\sigma^1}(E) - \phi^+_{\sigma^2}(E)|, |\phi^-_{\sigma^1}(E) - \phi^-_{\sigma^2}(E)| \right\} \in [0, \infty]$$

and a metric on the space $\mathrm{Stab}(\mathcal{D})$ of stability conditions

$$d(\sigma^1, \sigma^2) = \sup \left\{ d_S(\mathcal{P}, \mathcal{Q}), \|Z^1 - Z^2\| \right\}$$

Main Theorem 2.49. $\mathrm{Stab}(\mathcal{D})$ *is a smooth, finite-dimensional complex manifold such that*

$$Z\colon \mathrm{Stab}(\mathcal{D}) \longrightarrow \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbf{C})$$
$$(Z, \mathcal{P}) \longmapsto Z$$

is a local homeomorphism.

What the theorem is saying is that deformations in the target copy of $\mathbf{C}^n$ give a canonical lifting of the deformation $(Z, \mathcal{P}) \rightsquigarrow (W, \mathcal{P}^\#)$:
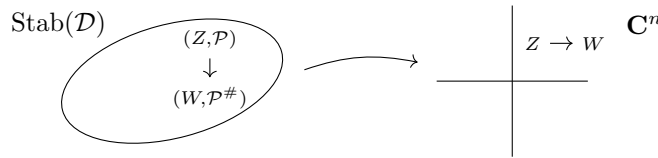
Example 2.50. Let Q be the Kronecker quiver

Choose $z_0, z_1 \in \mathbf{H}$, and $Z(V) = \dim(W_0)z_0 + \dim(W_1)z_1$. We get two cases based on the phase diagram relating z_0, z_1 as in Example 2.19: if $\phi(z_1) > \phi(z_0)$, then there are two stable objects S_0 and S_1, and if the opposite inequality holds, the space of stable objects will change.

In general, if $Z \rightsquigarrow W$ is a deformation of the central charge that is "small enough" so that the slicing does not change, and there exists a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

such that the orientation of the parallelogram with the phases of A, E, B changes under this deformation, i.e.,
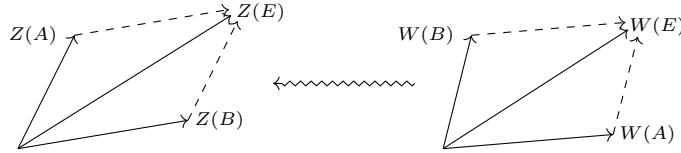
then the moduli space will change.

Example 2.51. Consider $\mathcal{A} = \mathsf{Rep}_Q$ the category of quiver representations for a quiver Q. Suppose $Q = (V_0, V_1)$ has $\#V_0 = n$. Define Z by choosing $z_1, \ldots, z_n \in \mathbf{H}$. Consider the deformation where z_i are unchanged for $i \neq n$, while z_n is reflected across the positive real axis: $z_n = x + i\epsilon \mapsto x - i\epsilon$, $x > 0$. Then,

$$\mathcal{P}((0, 1]) \rightsquigarrow \mathcal{Q}((0, 1]) = \mathcal{A}^{\#},$$

the tilt at the torsion pair $(\mathcal{T}, \mathcal{F})$ where $\mathsf{Obj}(\mathcal{F}) = S_n^{\oplus k}$ and $\mathsf{Obj}(\mathcal{T}) = \{V \in \mathsf{Obj}(\mathsf{Rep}_Q) \mid \mathrm{Hom}(V, S_n) = 0\}$. This example shows how the heart changes under wall-crossing.

We can also deform across the negative $\mathbf{R}$-axis, that is $z_n = -x + i\epsilon \mapsto -x - i\epsilon$, and so

$$\mathcal{P}((0, 1]) \rightsquigarrow \mathcal{Q}((0, 1]) = \mathcal{A}^{\#}[-1],$$

where $\mathcal{A}^{\#}$ is the tilt at the torsion pair $(\mathcal{T}, \mathcal{F})$, where $\mathsf{Obj}(\mathcal{T}) = \{S_n^{\oplus k}\}$ and $\mathsf{Obj}(\mathcal{F}) = \{V \mid \mathrm{Hom}(S_n, V) = 0\}$.

We now give some examples of stability manifolds.

Examples 2.52. If $\mathcal{D} = \mathsf{D}^b(\mathsf{Coh}(X))$ for X an algebraic variety,

1. $\dim X = 1$ is well understood: if $X = \mathbf{P}^1$, then $\mathrm{Stab}(\mathbf{P}^1) \cong \mathbf{C}^2$, and if $g(X) > 0$, then $\widetilde{\mathrm{SL}(2, \mathbf{R})} \circlearrowright \mathrm{Stab}(X)$, and
$$\mathrm{Stab}(X) \Big/ \widetilde{\mathrm{SL}(2, \mathbf{R})} \cong \{\mathrm{pt}\}$$

2. $\dim X = 2$: Bayer–Macri studied K3's, Xian Lei–Zhao studied $\mathbf{P}^2$, and abelian varieties are also understood.

3. $\dim X = 3$: There are partial results for abelian threefolds. On the other hand, open (e.g. toric) threefolds are understood.

3 Applications

We will give one application concerning Donaldson–Thomas and Pandharipande–Thomas invariants, which are numerical invariants of Calabi–Yau threefolds. We will follow the reference [Tod10].

3.1 History

Let X be a smooth projective Calabi–Yau threefold, where we recall that a threefold is *Calabi–Yau* if $\bigwedge^3 \Omega_X \cong \mathcal{O}_X$. An example of a Calabi–Yau threefold is the Fermat quintic $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$ in $\mathbf{P}^4$. In physics (the standard model), you can imagine that reality is 11-dimensional, in which case the inner space of strings are Calabi–Yau threefolds.

In order to give some predictions about reality, physicists associated some invariants to X. The first one historically was Gromov–Witten invariants.

Consider the moduli space of stable maps

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \left\{ \begin{array}{c} C \xrightarrow{f} X \\ C \text{ genus g} \end{array} \middle| \; f_*([C]) = \beta \in H_2(X) \right\}.$$

This is a Deligne–Mumford stack with infinitely many components that is very singular. From the symplectic point of view, the condition on the right is saying that the image of the curve has a fixed volume. In the special case where X is a Calabi–Yau threefold, we consider the *virtual fundamental class*

$$\left[\overline{\mathcal{M}}_{g,0}(X, \beta) \right]^{\mathrm{vir}},$$

which is a closed subscheme of the entire Deligne–Mumford stack. It is a dimension 0 scheme, so we can take the degree

$$\mathrm{GW}_{g,\beta} = \deg \left[\overline{\mathcal{M}}_{g,0}(X, \beta) \right]^{\mathrm{vir}} \in \mathbf{Q}$$

which is called the *Gromov–Witten invariant* of X. You can then put all values of this invariant for different g, β to give a generating function.

Example 3.1. When $g = 0$, then the generating function $J = \sum \mathrm{GM}_{0,\beta} t^\beta$ is a solution of a hypergeometric equation, the Picard–Fuchs equation (Givental, Liu–Lian–Yau).

For $g = 1$, Zinger spent ten years computing J; for $g \geq 2$, the question is wide open.

What people do instead of trying to compute Gromov–Witten invariants naïvely is to introduce new theories, and claim they are equivalent yet easier to understand than the older theories. For example,

- FJRW (Fan–Jarvis–Ruan–Witten) theory (Landau–Ginzburg/Calabi–Yau correspondence)
- DT (Donaldson–Thomas) theory (Donaldson–Thomas/Gromov–Witten correspondence)

We will be focusing on the latter of these.

3.2 Donaldson–Thomas invariants

Let X be a Calabi–Yau threefold, and consider the Hilbert scheme

$$I_n(X, \beta) = \left\{ \begin{array}{c} \text{subschemes } C \subset X, \; \dim C \leq 1 \\ [C] = \beta, \; \chi(\mathcal{O}_C) = n \end{array} \right\}$$

which we sometimes call the *Donaldson–Thomas moduli space*.

Property 3.2. $I_n(X, \beta)$ *is projective, and has a symmetric obstruction theory. We can therefore define* $I_{n,\beta} = \deg[I_n(X, \beta)]^{vir}$.

The reason why this obstruction theory is necessary is that general Hilbert schemes are very singular; to define intersection theory on this Hilbert scheme, then, requires a nicely behaved fundamental class. Obstruction theory allows us to define the *virtual fundamental class* which fills this role.

As was the case with the Gromov–Witten invariant, the virtual fundamental class $[I_n(X, \beta)]^{\mathrm{vir}}$ is such that $\dim[I_n(X, \beta)]^{\mathrm{vir}} = 0$. Heuristically, this virtual fundamental class "counts" curves inside X satisfying the cohomology condition $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$.

Like with Gromov–Witten invariants, we can put together information for different choices of n and β into a power series:

Definition 3.3. Let $I_{n,\beta} = \deg[I_n(X,\beta)]^{\mathrm{vir}} \in \mathbf{Z}$. Then, we define *Donaldson–Thomas theory* as

$$\mathrm{DT}(X) = \sum_{n,\beta} I_{n,\beta} x^n y^\beta.$$

Since this has some redundant information when $\beta = 0$, given by

$$\mathrm{DT}_0(X) = \sum_n I_{n,0} x^n,$$

we define *reduced Donaldson–Thomas theory* as

$$\mathrm{DT}'(X) = \frac{\mathrm{DT}(X)}{\mathrm{DT}_0(X)} = \sum_\beta \mathrm{DT}'_\beta(X) y^\beta.$$

It was open for a while exactly what this Laurent series looks like.

We now want to define Pandharipande–Thomas theory. One issue with Donaldson–Thomas theory was that it relied on the hypothesis that X was Calabi–Yau. Pandharipande–Thomas theory, on the other hand, can be defined even if X is not Calabi–Yau, which makes it useful for inductive proofs (since subvarieties of Calabi–Yau manifolds are seldom Calabi–Yau) and for proving that Donaldson–Thomas and Gromov–Witten theory are equivalent (this uses degeneration formulas).

Definition 3.4. We say (F, s) is a *stable pair* if F is a pure one-dimensional sheaf, and $s\colon \mathcal{O}_X \to F$ has cokernel of dimension zero. We can then define the moduli space

$$P_n(X,\beta) = \left\{ \begin{array}{c} \text{stable pairs } (F, s) \\ \text{with } [F] = \beta, \ \chi(F) = n \end{array} \right\} = \left\{ \mathcal{O}_X \xrightarrow{s} F \right\} \subset \mathsf{Obj}(\mathsf{D}^b(X))$$

which we sometimes call the *Pandharipande–Thomas moduli space*.

This looks similar to the Donaldson–Thomas moduli space from before.

Property 3.5. $P_n(X,\beta)$ *is projective, and has a symmetric obstruction theory. We can therefore define* $P_{n,\beta} = \deg[P_n(X,\beta)]^{vir}$.

Theorem 3.6. *The reduced Donaldson–Thomas invariant is the same as the Pandharipande–Thomas invariant, that is,* $\mathrm{DT}'(X) = \mathrm{PT}(X)$.

What is useful about this Theorem is that as we mentioned before, the left-hand side does not behave well under degeneration, but the right-hand side does. This is the key ingredient in showing Donaldson–Thomas and Gromov–Witten invariants are equivalent.

Idea of proof. Consider the derived category $\mathcal{D} = \mathsf{D}^b(\mathsf{Coh}\,X)$, and consider stability conditions $\sigma = (Z, \mathcal{P})$ on X. Recall from before that it is unknown whether the space of stability conditions on a Calabi–Yau manifold X is non-empty: the condition that $Z\colon K(\mathcal{D}) \to \mathbf{C}$ satisfies $Z(E) \in \mathbf{R}_{>0} e^{i\pi\phi}$ for $E \in \mathcal{P}(\phi)$ is the hardest to verify.

To remedy this situation, Toda invented what is called a *weak stability condition*:

Definition 3.7. Fix a finite rank abelian group Γ, and a filtration

$$0 \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_N = \Gamma.$$

Let $H_i = \Gamma_i / \Gamma_{i-1}$. Then, $\sigma = (\{Z_i\}, \mathcal{P})$ is a *weak stability condition* if
1. $\mathcal{P}$ is a slicing;
2. Each $Z_i\colon H_i \to \mathbf{C}$ is such that if we define for $v \in \Gamma$ that $Z(v) = Z_i([v])$ where $v \in \Gamma_i \setminus \Gamma_{i-1}$, then $Z(E) \in \mathbf{R}_{>0} \exp(i\pi\phi)$;
3. A condition on supports is verified.

Note the key difference from a stability condition is that Z does not have to be linear on all of $K(\mathcal{D})$. For weak stability conditions, we can show that their moduli space is nonempty by constructing one explicitly:

Example 3.8. Let X be a smooth projective variety, with $\dim X = d$, and W an ample divisor. Let $\mathcal{D} = \mathsf{D}^b(\mathsf{Coh}\,X)$, and let $\Gamma = \mathrm{im}(\mathrm{ch}\colon K(\mathcal{D}) \to H^*(X, \mathbf{Q}))$. Also, let $\Gamma_i = \Gamma \cap H^{\geq 2d-2i}(X, \mathbf{Q})$. Then, $H_i = \Gamma \cap H^{2d-2i}(X, \mathbf{Q})$. Choose $0 < \phi < \phi_{d-1} < \cdots < \phi_0 < 1$ and define Z_i so that

$$v \mapsto \exp(i\pi\phi_i) \underbrace{\int_X v \cup \omega^i}_{>0}$$

We then define
$$\mathcal{P}(\phi_i) = \{E \in \mathsf{Coh}\,X,\ E \text{ is pure of } \dim(\mathrm{Supp}) = i\}.$$

Now to prove the theorem, we define a subcategory $\mathcal{D}_X = \langle \mathcal{O}_X, \mathsf{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}}$, the smallest subcategory of $\mathsf{D}^b(\mathsf{Coh}\,X)$ containing $\mathcal{O}_X$ and all coherent sheaves with support of dimension ≤ 1. Toda then proceeded by finding the heart $\mathcal{A}_X$ of $\mathcal{D}_X$, which is a tilt of the standard t-structure on $\mathcal{D}_X$, and also constructed a family of stability conditions on $\mathcal{D}_X$, which depends on $(z_0, z_1) \in \mathbf{H}$. He showed that if the phase of z_1 is greater than that of z_0, then you get the Donaldson–Thomas moduli space, and if the phase of z_1 is smaller than that of z_1, then you get the Pandharipande–Thomas moduli space (this step is classical in flavor). Finally, he showed that this numerical invariant does not depend on stability conditions on a connected component, thus showing that $\mathrm{DT}' = \mathrm{PT}$. This last step is non-trivial, and requires studying the Hall algebra. $\square$

References

[BBD82]   Aleksandr A. Beĭlinson, Joseph N. Bernstein, and Pierre Deligne. "Faisceaux pervers." *Analysis and topology on singular spaces, I (Luminy, 1981)*. Astérisque 100. Paris: Soc. Math. France, 1982, pp. 5–171. MR: 751966.

[Bri07]   Tom Bridgeland. "Stability conditions on triangulated categories." *Ann. of Math. (2)* 166.2 (2007), pp. 317–345. ISSN: 0003-486X. DOI: 10.4007/annals.2007.166.317. MR: 2373143.

[Cra07]   Alastair Craw. *Explicit methods for derived categories of sheaves*. Lecture notes from *Derived Categories*, a VIGRE mini-course held June 4–16 at the University of Utah. 2007. URL: https://www.math.utah.edu/dc/tilting.pdf.

[Dou02]   Michael R. Douglas. "Dirichlet branes, homological mirror symmetry, and stability." *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*. Beijing: Higher Ed. Press, 2002, pp. 395–408. MR: 1957548.

[GM03]   Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Second ed. Springer Monographs in Mathematics. Berlin: Springer-Verlag, 2003. ISBN: 3-540-43583-2. DOI: 10.1007/978-3-662-12492-5. MR: 1950475.

[KS08]   Maxim Kontsevich and Yan Soibelman. *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*. Nov. 16, 2008. arXiv: 0811.2435 [math.AG].

[Tod10]   Yukinobu Toda. "Curve counting theories via stable objects I. DT/PT correspondence." *J. Amer. Math. Soc.* 23.4 (2010), pp. 1119–1157. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-10-00670-3. MR: 2669709.