Geometric Group Theory

(Selected highlights of δ-hyperbolic groups)

- (I) Basic philosophy
- (II) Quasi-isometries, Milnor–Švarc
- III δ-hyperbolicity, quasigeodesic stability
- IV The word problem for δ-hyperbolic groups (algorithmic properties of)
- V The Tits alternative for δ-hypebolic groups (structure of subgroups of)
- (VI) A smorgasbord

(I)

f.g. group G (discrete) $\curvearrowright$ X proper, geodesic metric space

properly discretely, isometrically (cocompactly)

various notions coincide for X locally cpt, Hausdorff:

(i) $\forall x \in X\ \exists U \ni x$ nbd s.t. $\#\{g \in G \mid g\cdot x \in U\} < \infty$

(ii) $(g, x) \mapsto (x, g\cdot x)$ is proper

(iii) $\forall \underset{\text{cpt}}{K} \subset X$: $\#\{g \in G \mid gK \cap K \neq \phi\} < \infty$

POINT: $X \to X/G$ is a covering map, and X/G is Hausdorff

$d(g\cdot x, g\cdot y) = d(x, y)\ \forall g \in G\ \forall x, y \in X$.

X/G compact

proper: closed balls are compact ($\Leftrightarrow$ bdd, in the metric space case.) Prevents co-dim'l nonsense

geodesic: Shortest paths exist / distance given by length of shortest path. Gives us a sensible notion of distance.

(Exercise A Riemannian manifold is proper and geodesic $\Leftrightarrow$ it is complete.)

~~Prop~~ $G \overset{\text{p.d.}}{\underset{\text{cocpt}}{\curvearrowright}} X \Rightarrow G$ f.g.

Idea of pf Take $D = B_{2\,\mathrm{diam}\,X/G}$

$S = \{g \in G \mid G(D) \cap D \neq \phi$

(e.g. $\pi_1(X) \curvearrowright \tilde{X}$

$G \curvearrowright \mathrm{Cay}(G, S)$

$\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$, $\mathrm{Out}(F_n) \curvearrowright CV_n$, $G_{\text{cubulated}} \curvearrowright$ CAT(0) cube complex ...)

<u>Basic philosophy</u> Use geometry / topology of space to study properties of group and vice versa.

<u>e.g.</u> (topological group theory) ~~Thm $F_n \cong \pi_1(\underbrace{S_1 \vee \dots \vee S_1}_{n \text{ circles}})$~~

<u>Thm</u> Subgroups of free groups are free.

<u>Sketch of pf</u> $F_n \cong \pi_1(\underbrace{S_1 \vee \dots \vee S_1}_{n \text{ circles}})$

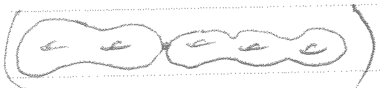
$$\pi_1(X) \longleftrightarrow \tilde{X}, \qquad H \le \pi_1(X) \longleftrightarrow \tilde{X}/H$$

$H \le F_n \Rightarrow H \cong \pi_1(\text{cover of } S_1 \vee \dots \vee S_1) = \pi_1(\text{quotient of } \infty\ 2n\text{-valent tree})$

$= \pi_1(\text{graph}) = \text{free group}$

Can generalize this: <u>Kurosh Subgroup Thm</u> $H \leq G_1 * G_2 \Rightarrow H = F * \underset{\lambda \in \Lambda}{*} H_\lambda$

free; each conjugate to subgroup of G_1 or G_2

e.g. $H \leq \pi_1$ ()
$\quad = \pi_1$ (tree of hyperbolic ~~sfcs~~ covering )

II Groups as geometric objects:

given G a f.g. group, S a finite generating set $(S^{-1} = S)$

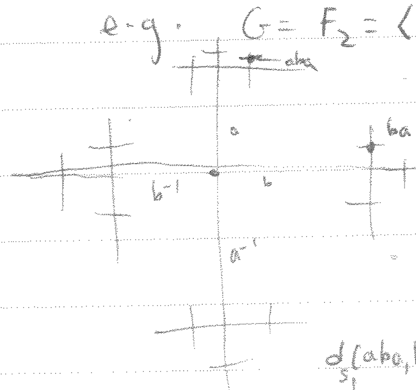
form <u>Cayley graph $Cay(G, S)$</u> by:

- {vertices} = G
- $g - h$ is an edge (labelled s) $\overset{def}{\Longleftrightarrow} \exists\, s \in S$ s.t. $gs = h$

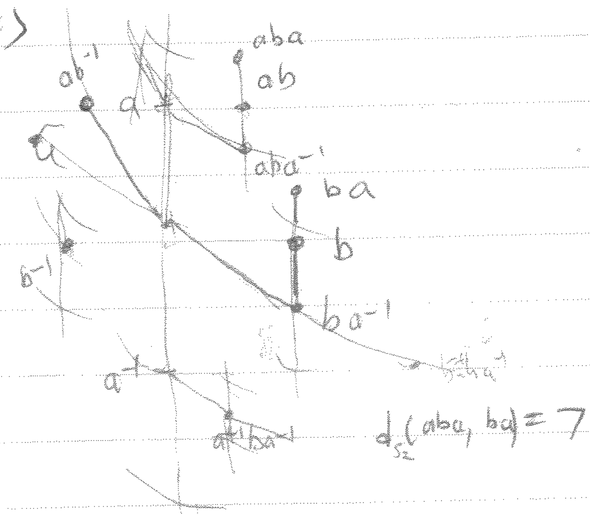
Put <u>word metric</u> on $Cay(G, S)$ by declaring all edges to have length 1.

(!) S not unique / canonical $\Longrightarrow$ no canonical word metric on G.

e.g. $G = F_2 = \langle a, b \rangle = \langle a, ba^{-1} \rangle$



$d_{S_1}(aba, ba) = 5$

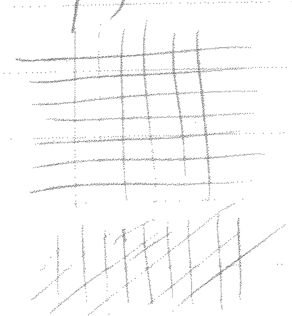


$d_{S_2}(aba, ba) = 7$

<u>BUT</u> "$Cay(G)$" / word metric on G is well-defined up to a natural equivalence relation (quasi-isometry)

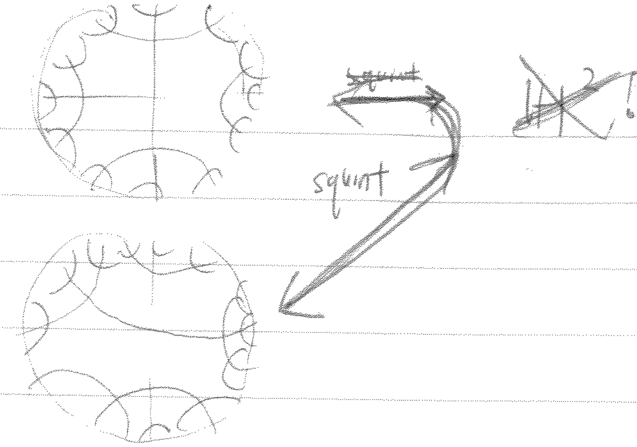
<u>Idea</u> $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$

$= \langle b, ab \mid [a, ab] \rangle$



Squint $\rightarrow$ both kind of look like $\mathbb{R}^2$.

or $\mathbb{F}_2 = \langle a, b \rangle$



<u>Defn</u> $h: X \to Y$ is a <u>quasi-isometric embedding</u> if $\exists\, k, C > 0$ s.t.

$$\frac{1}{k} d(x,z) - C \le d(h(x), h(z)) \le k\, d(x,z) + C \quad \forall x, z \in X$$

h is a <u>quasi-isometry</u> (q.i.) if it is also coarsely surjective, i.e. $\exists\, D \ge C$ s.t. $\forall y \in Y$: $\exists\, x \in X$ s.t. $d(h(x), y) \le D$.

<u>e.g.</u> $Cay(\mathbb{Z}^2, \{a,b\}) \hookrightarrow \mathbb{R}^2$ is a $(\sqrt{2}, \frac{1}{2})$-quasi-isometry

$Cay(\mathbb{Z}^2, \{b, ab\}) \hookrightarrow \mathbb{R}^2$ is a $(2, \frac{\sqrt{2}}{2})$-q.i.

~~$Cay(F_2, \{a,b\}) \hookrightarrow \mathbb{H}^2$ is a $(2\cosh^{-1}(1), \frac{1}{2})$-q.i.~~ oops
?

<u>Prop</u> A composition of q.i.'s is a q.i.

<u>Prop</u> If $h: X \to Y$ is a q.i., then $\exists\, j: Y \to X$ a q.i. s.t. $\exists R \ge 0$: $d(j(h(x)), x)$, $d(h(j(y)), y) \le R \quad \forall x \in X, y \in Y$.

j is called a <u>quasi-inverse</u>

<u>Pf (idea)</u> Choose $j(y)$ to be (any) point in $h(X)$ closest to y.

<u>UPSHOT</u> quasi-isometry is an equivalence relation between (proper, geodesic) metric spaces

<u>Prop.</u> $Cay(G, S) \overset{q.i.}{\sim} Cay(G, S')$ for any two finite generating sets S, S'.

<u>Pf</u> Let k_1 = max word length of elt of S' in terms of S

~~k_2 = " S " S'~~

Map $s \in S$ onto geodesic path in $Cay(G, S')$. Then we have

$$d_S(g,h) \le d_{S'}(g,h) \le k_1 d_S(g,h)$$

p1 ↑
p2 ↓

Lemma (Milnor-Švarc) $G \underset{\text{cocptly}}{\curvearrowright} X \Rightarrow G$ with any word metric is qi to X.

(**Cor.** If $G \underset{\text{cocptly}}{\curvearrowright} X$, $G \underset{\text{cocptly}}{\curvearrowright} Y$, then $X \overset{qi}{\simeq} Y$.)

Pf We will prove $\overset{\text{(any)}}{\text{the}}$ orbit map is a qi:

fix $x_0 \in X$, let $\tau : G \to X$ be given by $\tau(g) = g(x_0)$.

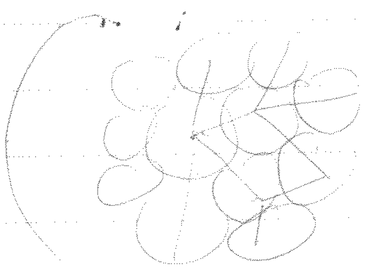
Extend this to a map $Cay(G, S) \to X$.

(By cocompactness) τ is coarsely $\operatorname{diam}(X/G)$-surjective

To get upper bound $d(\tau(g), \tau(h)) \leq K d_S(g,h) + C$

$$\text{Let } Q = \max_{s \in S} d(x_0, s x_0).$$

$$d(\tau(h), \tau(hg)) = d(\tau(1), \tau(g)) \leq Q\, d_S(1, g) = Q\, d_S(h, hg)$$



$\forall\, h, g \in G$.

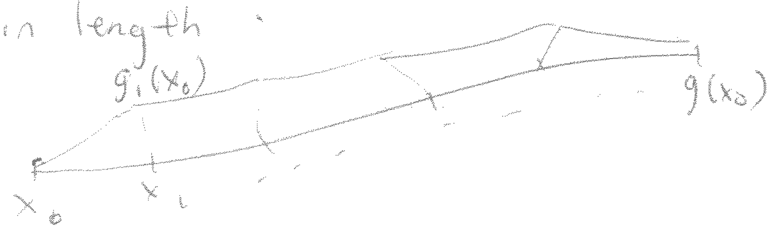
$\cancel{d(g,h) \leq k\, d(\tau(g), \tau(h)) + C}$

To get lower bound $\boxed{\cancel{d(\tau(g), \tau(h)) \geq \frac{1}{k} d(g,h) - C}}$ (✓)

Write $D = \operatorname{diam}(X/G)$. Let $S' = \{g \in G \mid d(x_0, g \cdot x_0) \leq 3D\}$, $P = \max_{s \in S'} d_S(1, s')$.

~~Will write any $g \in G$ as s~~ (Show any path from x_0 to $g \cdot x_0$ goes through ~~Will~~ at least $\cancel{\frac{d_P(1,g)}{P}}$ segments $\leq D$ in length.)

Divide geodesic $[x_0, g(x_0)]$ into $N := \left\lfloor \frac{d(x_0, g x_0)}{D} \right\rfloor + 1$ segments each $\leq D$ in length.



For each $i = 1, \dots, N$, choose $g_i \in G$ s.t. $d(x_i, g_i(x_0)) \leq D$,

$g_0 = id$, $g_N = g$.

Then $d(g_i(x_0), g_{i+1}(x_0)) \leq 3D$

$\qquad \| \qquad\qquad\qquad\qquad \Rightarrow g_i^{-1} g_{i+1} \in S'$

$\quad d(x_0, g_i^{-1} g_{i+1}(x_0))$

$$g = g_0 (g_0^{-1} g_1)(g_1^{-1} g_2)(g_2^{-1} g_3) \cdots (g_{N-1}^{-1} g_N)$$

$$d_S(1, g) \leq NP \leq P\left(\frac{d(x_0, g x_0)}{D} + 1\right)$$

$$\Rightarrow \frac{D}{P} d_S(1, g) - D \leq d(x_0, g x_0).$$

III

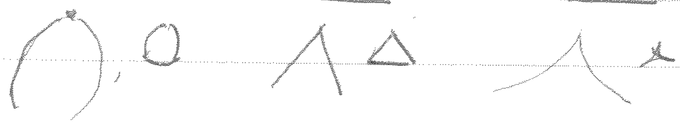
Recall: ~~Milnor-Švarc~~ $F_2 \overset{q.i.}{\sim} \mathbb{H}^2$

(Milnor–Švarc) $\pi_1(\Sigma_g) \overset{q.i.}{\sim} \mathbb{H}^2$ $(g \geq 2)$

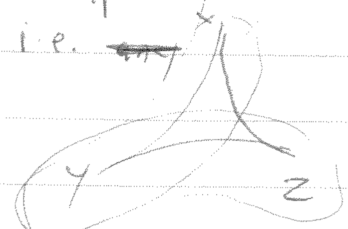
In geometry, negative curvature is often very useful.

What is the analogue of $K<0$ here?

In Riemannian geometry: $K>0$ $K=0$ $K<0$

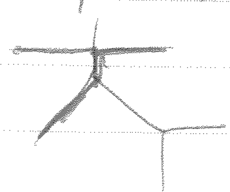


Def^n (Gromov) X is δ-hyperbolic if geodesic triangles in X are δ-slim, i.e. any point on $\overline{xz}$ is in a δ-nbd of $\overline{xy} \cup \overline{yz}$.



($\iff$ ε-thin, i.e. geodesic triangles are contained in ε-thickenings of corresponding tripods, but this is not the same def^n. For equivalence: exercise, or see Bridson and Haefliger, Chap III.H, Prop. 1.17.)

e.g. (metric) trees are 0-hyperbolic:



$\mathbb{H}^2$ is $\cosh^{-1}(2)$-hyperbolic

exercise! find best constant

Def^n G is hyperbolic if it acts cocompactly on some δ-hyperbolic metric space.

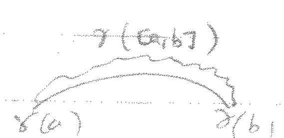
e.g. $\pi_1(\Sigma_g) \curvearrowright \Sigma_g$ $(g>1)$, $F_n \curvearrowright$ tree, (all finite groups), ...

Morse Lemma (quasigeodesic stability)

(Def^n A (k,c)-quasigeodesic (embedding) is the image of a geodesic under a (k,c)-qi)

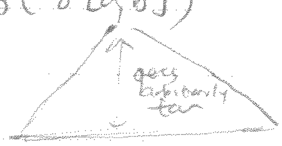
Let $\gamma : [a,b] \to X$ (δ-hyp) be a (k,c)-quasigeodesic. Then $\exists\, D = D(k,c,\delta)$ s.t.

$$\gamma([a,b]) \subset \mathcal{N}_D([\gamma(a),\gamma(b)]), \quad [\gamma(a),\gamma(b)] \subset \mathcal{N}_D(\gamma[a,b])$$



NOTE Same is not true in $K \geq 0$!

also log spirals are quasigeodesics in $\mathbb{E}^2$.

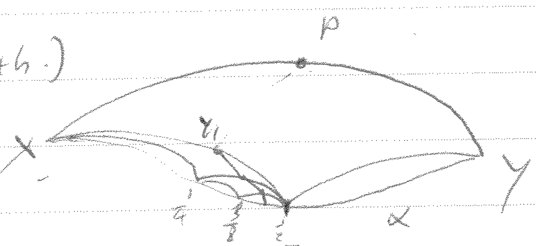


Pf ~~Lemma (exponential divergence of geodesics)~~ ... even before that:/

Technical Lemma: Arbitrary (k,c)-quasigeodesics are $D'=D'(k,c)$ close to rectifiable (k',c')-quasigeodesics, where $(k',c')=f(k,c)$.

Lemma (exponential divergence of geodesics) X δ-hyperbolic, $p\in[x,y]\subset X$, $\alpha:[0,1]\to X$ rectifiable path joining x to y. Then
$d(p,\alpha([0,1]))\le \delta\log_2(\ell(\alpha))+1$.

pf (Parametrize α prop. to arclength.)



choose $N\in\mathbb{N}$ s.t. $2^N\le\ell(\alpha)<2^{N+1}$
($so\ 1\le\ell(\alpha[\frac{k}{2^N},\frac{k+1}{2^N}])<2$)

Consider geodesic Δ $[x,y]\cup[x,\alpha(\frac12)]\cup[\alpha(\frac12),y]$.

Pick $y_1\in_{\text{wlog}}[x,\alpha(\frac12)]$ s.t. $d(p,y_1)<\delta$.

Consider Δ $[x,\alpha(\frac12)]\cup[x,\alpha(\frac14)]\cup[\alpha(\frac14),\alpha(\frac12)]$

Pick $y_2\in_{\text{wlog}}[\alpha(\frac14),\alpha(\frac12)]$ s.t. $d(y_1,y_2)<\delta$

... continue until we get $y_N\in[\alpha(\frac{k}{2^N}),\alpha(\frac{k+1}{2^N})]$
s.t. $d(y_N,y_{N-1})<\delta$.

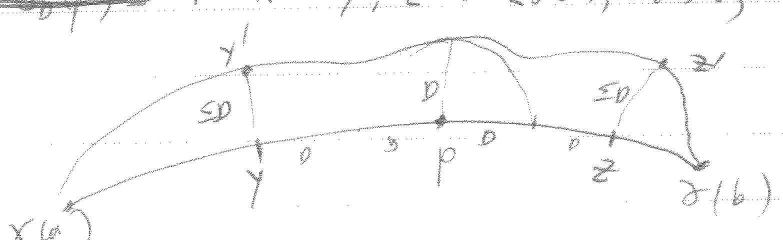
Then $d(p,\alpha([0,1]))\le d(p,y_N)+d(y_N,\alpha([0,1]))$
$<\delta N+1$
$\le\delta\log_2(\ell(\alpha))+1$. ∎

To show $\gamma([a,b])\subset\mathcal{N}_D([\gamma(a),\gamma(b)])$ for $D=D(k,c,\delta)$

Let $D=$ Hausdorff distance from $\gamma([a,b])$ to $[\gamma(a),\gamma(b)]$, (max pointwise) i.e. smallest D s.t. containment holds

$p\in[\gamma(a),\gamma(b)]$ be s.t. $d(p,\gamma)=D$.

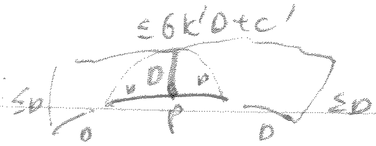
~~Let $B_D(p)$~~ Pick $y,z\in[\gamma(a),\gamma(b)]$, $y',z'\in\gamma([a,b])$ as shown



$d(y,y'),d(z,z')\le D \implies d(y',z')\le 6D$

$\implies \ell(\gamma([y',z']))\le 6k'D+c'$

Exponential divergence of geodesics

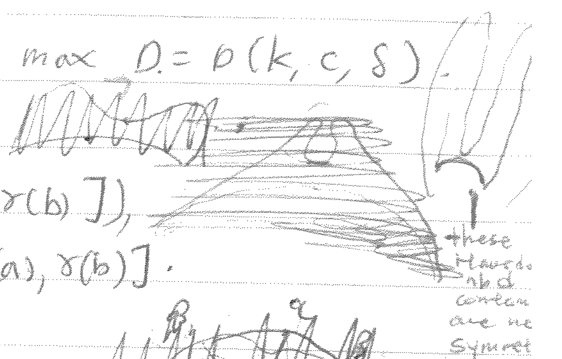


$$d(p, \text{red}) = D \leq \delta \log_2 (6k'D + c' + 4D) + 1$$

~~thus~~ logs are dominated by linears

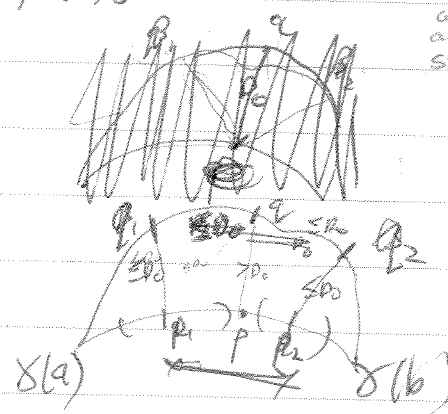
$\Longrightarrow$ inequality only true up to some max $D_0 = D(k, c, \delta)$.

To show $[\gamma(a), \gamma(b)] \subset \mathcal{N}_{D_0}(\gamma[a,b])$



Pick $q \in \gamma([a,b])$ maximizing $d(q, [\gamma(a), \gamma(b)])$,
let p be corresponding furthest point on $[\gamma(a), \gamma(b)]$.

If $D_q := d(q,p) \leq \underset{\text{from above}}{D_0}$, done



Else: Choose $q_1, q_2 \in \gamma([a,b])$ s.t.
$d(q, q_i) \leq D_0$. ~~Then γ btw q_1, q_i~~
~~has length $\leq k'D_0 + c'$~~

~~Choose p_1~~

Choose $p_1, p_2 \in [\gamma(a), \gamma(b)]$ s.t. ~~$d(q_1, p_1), d(q_2, p_2)$~~ $d(p_i, q_i) \leq D_0$
and $p \in \overline{p_1 p_2}$.

Then $d(p_1, p_2) \leq 4D_0$, so $D_1 \leq 2D_0 + 2D_0 = 4D_0$. $\square$

$\boxed{\text{Corollary}}$ δ-hyperbolicity is qi-invariant (i.e. X δ-hyperbolic,
$f: X \to Y$ a qi $\Rightarrow$ Y δ'-hyperbolic (where $\delta' = \delta'(k, c, \delta)$).)

1 D 2
2 D 3

IV Recall: X is δ-hyperbolic if it has δ-slim triangles

Defn A f.g. group G is δ-hyperbolic if it acts cocompactly on some δ-hyperbolic metric space (WLOG, can take space to be $Cay(G,S)$ by Milnor–Švarc)

e.g. $\pi_1(\Sigma_g) \curvearrowright \Sigma_g$ $(g>1)$, $F_n \curvearrowright$ tree, (all finite groups), ...
$Mod(S) \curvearrowright \mathcal{C}(S)$
$Out(F_n) \curvearrowright \mathcal{F}_n$

Today: δ-hyperbolic groups have good algorithmic properties
e.g. • word problem solvable in linear time (are automatic)
• conjugacy problem solvable (have decidable marked isomorphism problem)

<u>Word problem</u> Given $G = \langle S \mid R \rangle$, w a word in elts of S, decide if $w \sim id_G$

<u>One way to solve this</u> <u>Defn</u> A finite presentation $\Gamma = \langle S ; R \rangle$ is a <u>Dehn presentation</u> if whenever w is a word in S s.t. $w \sim id_G$, $\exists$ subword w_0 of w which is "more than $\frac{1}{2}$ a relator"

e.g. ($aba^{-1}b^{-1}$ relator, aba^{-1} "$> \frac{1}{2}$") ($abcaba^{-1}$: relator, bca more than half)

i.e. $\exists$ word v_0 s.t. $\ell(v_0) < \ell(w_0)$, $v_0 w_0 \in R$.

<u>Prop.</u> If G has a Dehn presentation, then the word problem in G is solvable in linear time

<u>Pf</u> Consider the algorithm:

(1) Given w, search w for subwords which are $> \frac{1}{2}$ a relator

(2) If none, $w \not\sim id_G$.

(3) Else: have w_0 s.t. $v_0 w_0 \in R$, $\ell(v_0) < \ell(w_0)$
Write $w = a w_0 b \sim a v_0^{-1} b =: w'$. $\ell(w') < \ell(w)$.

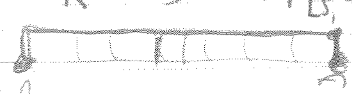
(4) If $w' = id$, win. Else, repeat.

This terminates in $\leq \ell(w)$ steps.

(linear in length of word)
(* implied constant/s depend on # relators, length of relators, performance of substring search.)

<u>Thm</u> Hyperbolic groups have Dehn presentations.
(In particular, they are finitely presented, and have solvable word problem)

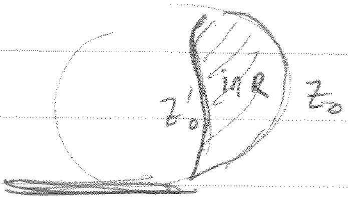
<u>Pf</u> Let A be <u>any</u> finite generating set, ~~R be~~
R be $\{$words w in $A \mid w \sim id_G, \ell(w) < 16\delta + 2\}$.
We claim $\langle A \mid R \rangle$ is a Dehn presentation.

<u>Key Lemma</u> (<u>Defn</u> A k-<u>local geodesic</u> is a path $\alpha : [a,b] \to X$ s.t. $s, t \in [a,b]$ with $|s-t| < k \Rightarrow \alpha|_{[s,t]}$ geodesic.)
(NOTE: not necessarily geodesics, e.g.  is a $(\frac{n}{2})$-local geodesic)

If X is δ-hyperbolic, $k > 8\delta$, then non-constant k-local geodesics in X are not closed loops.

Let $z \sim id$ be a word in A (a closed loop based at $id_G \in Cay(G, A)$). By Key Lemma, z is not a $(8\delta+1)$-local geodesic

$\Longrightarrow \exists$ subword Z_0 of length $\le 8\delta+1$ which is not a geodesic.
Let Z_0' be a geodesic joining endpoints of Z_0. $\ell(Z_0') < \ell(Z_0)$



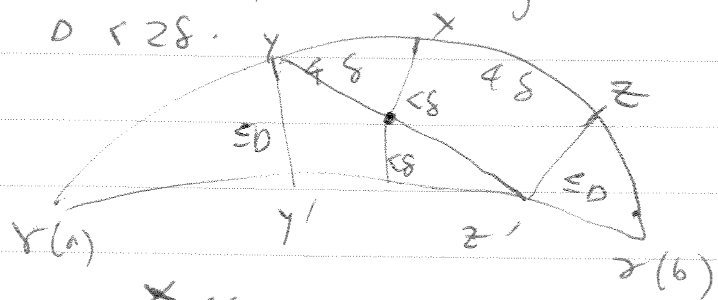
Now $Z_0(Z_0')^{-1} \sim id$, $\ell(Z_0(Z_0')^{-1}) < 2(8\delta+1) = 16\delta+2$

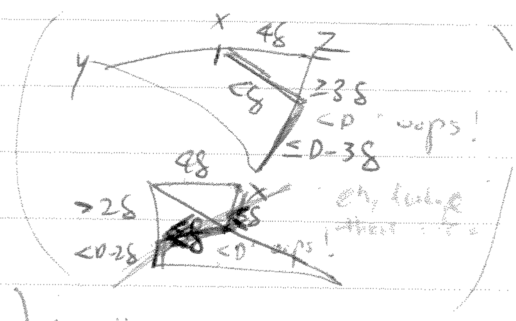
$\Longrightarrow Z_0(Z_0')^{-1} \in R$

$\Longrightarrow Z$ has a subword which is $> \frac{1}{2}$ a relator in R $\square$

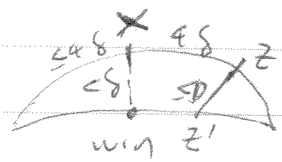
<u>Sketch of Pf of Lemma</u> ~~[illegible]~~ (1) γ a k-local geodesic
$\Longrightarrow \gamma([a,b]) \subset N_{2\delta}([\gamma(a), \gamma(b)])$.

<u>Pf</u> Choose $x \in \gamma([a,b])$ maximizing $D := d(x, [\gamma(a), \gamma(b)])$.
WTS $D < 2\delta$.

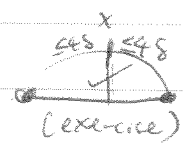




(OR



OR



) ← "exercises"

(In fact, – $[\gamma(a), \gamma(b)] \subset N_{2\delta}(\gamma([a,b]))$
– γ is a $\left(\frac{k+4\delta}{k-4\delta}, 2\delta\right)$ – quasigeodesic
("k-local geodesics make roughly linear progress"))

(2) If $|a-b| \le k$, γ a geodesic $\Rightarrow$ not a nontrivial closed loop
(3) Else (if $|a-b| \ge k > 8\delta$) $\gamma|_{[a,a+4\delta]}$ is a geodesic, so $d(\gamma(a), \gamma(a+4\delta)) = 4\delta$
~~But~~ If γ were a closed loop, then $N_{2\delta}([\gamma(a), \gamma(b)]) = B_{2\delta}(\gamma(a))$
$\not\ni \gamma(a+4\delta)$ ↯

~~in linear time~~

<u>Remark</u> • Conjugacy problem solvable, for δ-hyperbolic groups, ~~[illegible]~~
• they contain finitely many conjugacy classes of finite-order elts — this can also be proven using Dehn presentations.

<u>Remark</u> In fact, G has a Dehn presentation <u>iff</u> it is δ-hyperbolic.

V ~~What can we say abt~~ So far $G \curvearrowright X$; δ-hyperbolic spaces and groups;

δ-hyperbolic groups have good algorithmic properties.

Today Subgroups of δ-hyperbolic groups satisfy the Tits Alternative

Thm G δ-hyperbolic, $\Gamma \leq G \Rightarrow$ OOTFH

(1) Γ virtually cyclic

(2) Γ contains a free group of rank 2

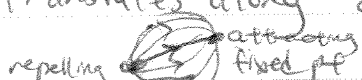
(cf. Thm (Tits, 1970's) $\Gamma \leq SL_n(\mathbb{R})$, OOTFH: (1) Γ virtually solvable; (2) $\Gamma \geq F_2$.)

(Outline of) proof

For $PSL_2\mathbb{R} \cong \mathrm{Isom}^+(\mathbb{H}^2)$:

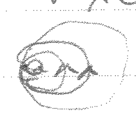
(1) Any $\gamma \in PSL_2\mathbb{R}$ is conjugate to one of

"hyperbolic" (a) $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$ — fixes pair of points in $\partial\mathbb{H}^2$, translates along geodesic "axis" between them by $2\log\lambda$



"north-south dynamics"

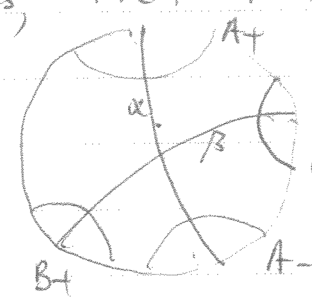
"parabolic" (b) $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ — fixes point in $\partial\mathbb{H}^2$, flowlines are horocycles



"elliptic" (c) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ — fixes point in $\mathbb{H}^2$

(2) **Lemma** If Γ contains 2 hyperbolic elements with distinct fixed point sets, then $\Gamma > F_2$.

Idea of Pf Ping-pong



$\alpha((A^{\pm})^c) \subset A^{\mp}$

$\beta((B^{\pm})^c) \subset B^{\mp}$

(3) If $\Gamma \not\ni$ 2 such hyperbolics, show and get virtual solvability.

For δ-hyperbolic groups

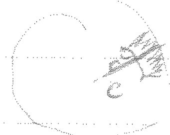
① To get similar classification of elts of $\mathrm{Isom}(\underset{\delta\text{-hyperbolic}}{X})$, we will develop notion of $\partial_\infty X$:

<u>Defn</u> $\partial_\infty X$ (the Gromov boundary or visual boundary of X) is defined as $\{\underset{\text{(by Morse lemma)}}{\underset{\text{quasigeodesic}}{\text{geodesic}}}\text{ rays}\}/\text{bdd Hausdorff distance}$

with topology of convergence $\left(\underset{\text{geodesic segments}}{[\alpha_n]} \to \underset{\text{geodesic ray}}{[\alpha]} \overset{\text{def}}{\iff} \underset{X}{x_n \to x \in \partial X}\right)$, in compact-open topology

and closed sets in $\overline{X} = X \cup \partial X$ contain all of their limit points.

(or defined using basis $\{B(x,r) \subset X\} \cup \left\{x \in \overline{X} \;\middle|\; \begin{matrix} d(x, c(0)) > r, \\ d(\rho_r(x), c(r)) < \varepsilon \end{matrix}\right\}$)



<u>Key properties</u>

- <u>Visibility</u>: ① $\forall p \in X$, $z \in \partial X$: $\exists$ geodesic ray α s.t. $[\alpha] = z$, $\alpha(0) = p$

(Key ingredients in proof: Arzelà–Ascoli, δ-hyperbolicity)

② If $z_1 \neq z_2 \in \partial X$, $\exists$ geodesic $\gamma: \mathbb{R} \to X$ s.t. $[\gamma|_{[0,-\infty)}] = z_1$, $[\gamma|_{[0,+\infty)}] = z_2$.

not nec. unique!

- <u>Bordification</u>: $X \cup \partial X$ is compact.
- <u>Extension of maps</u>: $f: X \to Y$ qi extends to $\partial f: \partial X \to \partial Y$ a bi-Lipschitz homeo (given by $\partial f([\alpha]) = [f \circ \alpha]$.

(Corollary: $\partial_\infty \Gamma$ well-defined for Γ a δ-hyperbolic group)

Now we can classify our isometries...

<u>Defn</u> An isometry γ of $\underset{\delta\text{-hyp.}}{X}$ is...

- ... <u>elliptic</u> if some orbit $\{\gamma^n(x_0)\}$ is bounded
- ... <u>parabolic</u> if some orbit $\{\gamma^n(x_0)\}$ has exactly one accumulation point
- ... <u>hyperbolic</u> if $n \mapsto \gamma^n(x_0)$ is a q-i embedding for some (and hence all) x_0.

~~(q-i group ...)~~ other we can take ...

Fact γ not elliptic or parabolic $\Rightarrow$ γ hyperbolic, exhibits N-S dynamics on $\overline{X}$, i.e. $\exists$ fixed points $\partial_\pm\gamma \in \partial_\infty X$ s.t. $\gamma^{\pm n}(z) \to \partial_\pm \gamma$ $\forall\ z \in \overline{X} \setminus \{\partial_\mp \gamma\}$, uniformly on compact subsets of $\overline{X} \setminus \{\partial_\mp\gamma\}$.

(**Lemma** If $\{\gamma^n(z_0)\}$ accumulates at $a \in \partial_\infty X$, then $\partial\gamma(a) = a$.)

Fact γ hyperbolic $\Rightarrow$ $\exists\, m > 0$ s.t. γ^m has an invariant geodesic axis (not necessarily unique) ($\uparrow$ $m > 1$, possibly, for some reason)

Proofs involve lots of sequences / convergence arguments, quasigeodesic stability and δ-slim triangles.

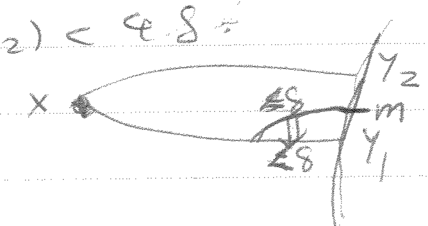
② δ-Hyperbolic ping-pong.

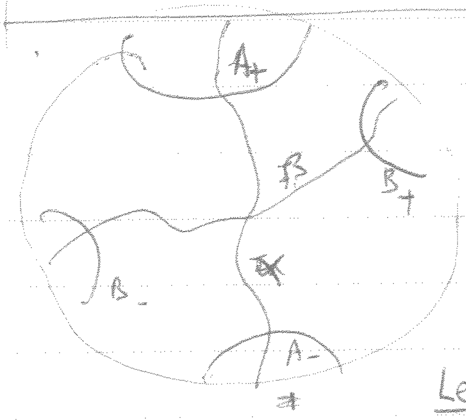
Lemma Given Γ δ-hyperbolic, $\gamma_1, \gamma_2 \in \Gamma$ hyperbolic w/ distinct fixed point sets (in $\partial_\infty \Gamma$). Then $\exists\, N > 0$ s.t. $\langle \gamma_1^N, \gamma_2^N \rangle \cong F_2$.

Pf **Lemma** (Projection onto geodesics is coarsely well-defined in δ-hyp. spaces)

Given $\underset{\text{geodesic}}{I} \subset \underset{\delta\text{-hyp}}{X}$, $x \in X$ and $y_1, y_2 \in I$ s.t. $d(y_1, x) = d(y_2, x) = d(x, I)$, we have $d(y_1, y_2) < 4\delta$.

Sketch of Pf





Define $A_+ := \{x \in \mathrm{Cay}(\Gamma) \mid \exists\, t_i > R : d(x, \alpha) = d(x, \alpha(t_i))\}$

i.e. x (coarsely) projects to $> R$ part of invariant axis

sim. $A_- := \{x \in \mathrm{Cay}(\Gamma) \mid \exists\, t_i < -R : d(x, \alpha) = d(x, \alpha(t_i))\}$

and for $B_\pm$

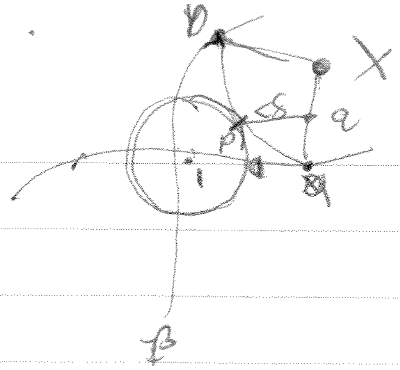
Lemma $\Rightarrow$ $R > 4\delta$ suffices to separate $A_\pm$.

To separate $A_\pm$ from $B_\pm$: let $\hat{\rho} = \max\{d(\alpha(0), 1), d(\beta(0), 1)\}$

such ρ exists by δ-hyperbolicity: $\rho > 0$ be s.t. $[a, b] \cap B_\rho(1) \neq \emptyset$ whenever $a \in \alpha$, $b \in \beta$

Choose $R > 2\rho + 2\hat{\rho} + 4\delta$.

Then:



Now ~~$d(q, \alpha(0)) = d($~~ $d(a,q) \le d(q, \alpha(0))$

$\le d(a,p) + d(p,1) + d(1,\alpha(0))$

(if $q \in \overline{ax}$) $\le \delta + \rho + \hat{\rho}$.

and so $d(a, \alpha(0)) \le d(a,q) + d(q, \alpha(0)) \le 2(\delta + \rho + \hat{\rho}) < R$

Similarly, ($d(b, \beta(0)) < R$ if $q \in \overline{bx}$) ~~Now $b = \beta(t)$ for some t~~

So at least one of $d(b, \beta(0)) < R$, $d(a, \alpha(0)) < R$ always holds, i.e. by choosing $R > 4\delta + 2\rho + 2\hat{\rho}$, $A_\pm$, $B_\pm$ are pairwise disjoint.

Finally (phew) choose n s.t. $nM\tau(\alpha)$, $nM\tau(\beta) > 3R$ and $N = nM$

(Def2 $\tau_{\Gamma,S}(\gamma) := \lim_{n \to \infty} \frac{d_{\Gamma,S}(1, \gamma^n)}{n}$ is the stable translation length)

(It is well-defined since $n \mapsto d_{\Gamma,S}(1, \gamma^n)$ is subadditive, so ← Fekete's lemma

$\frac{f(n)}{n} = \frac{f(dq+r)}{dq+r} \le \frac{q f(d) + f(r)}{dq+r} \le \frac{f(d)}{d} + \frac{f(r)}{r} \Rightarrow \overline{\lim} \frac{f(n)}{n} \le \underline{\lim} \frac{f(d)}{d}$.)

Suppose $x \in Cay(\Gamma) \setminus A_-$, let $\alpha(t)$ be a projection of x onto α

~~$d(\alpha^{nM}(x), \alpha) = d(\alpha(t + nM\tau(\alpha)), \alpha)$ $d(\alpha^{nM}(x), \alpha(t + nM\tau(\alpha)))$~~

~~$\Rightarrow \alpha(t + nM\tau(\alpha))$~~ $d(x, \alpha) = d(x, \alpha(t))$

$d(\alpha^{nM}(x), \alpha) = d(\alpha^{nM}(x), \alpha(t + nM\tau(\alpha)))$

$\Longrightarrow \alpha(t + nM\tau(\alpha))$ is a projection of $\alpha^{nM}(x)$ onto α.

Since $x \notin A_-$, $t \ge -R$, so $t + nM\tau(\alpha) > 2R$,

so any projection of $\alpha^{nM}(x)$ onto α has coordinate $\ge 2R - 4\delta > R$

so $\alpha^{nM}(x) \in A_+$ ("morally" projections not just coarsely well-defined, but also coarsely equivariant.)

$\Longrightarrow \alpha^N(Cay(\Gamma) \setminus A_-) \subset A_+$

Similarly $\alpha^{-N}(Cay(\Gamma) \setminus A_+) \subset A_-$, and for $B_\pm$ with $\beta_\pm$.

Then play ping-pong as before. □

Pf of Tits Alternative for δ-hyperbolic groups

(3) • Γ finite $\Rightarrow$ (trivially) win.

• Γ infinite $\Longrightarrow$ Γ has an ∞-order element

nice argument with ∂X.

– If enough such elts, play ping-pong and get F_2.

– else: Claim α, β ∞-order, share one fixed point

$\Rightarrow \alpha, \beta$ share both fixed points. (if works)

Claim All ∞-order elts of Γ share both fixed points $\Rightarrow \Gamma$ virtually cyclic

Pf (Idea) WLOG (up to an index-2 subgroup) Γ preserves fixed points.

$\exists$ finitely many (say, n) possible invariant axes; take $\phi: \Gamma \to S_n$

and $\Gamma_1 = \ker \phi$. This is a f.i. subgroup of Γ,

and cyclic since $\Gamma_1 \curvearrowright \underset{\text{any axis}}{I}$ freely, properly. ∎

(Idea for pf of previous claim: Consider conjugates $\alpha^{-n}\beta\alpha^n$. These $\xrightarrow{\text{(WLOG)}} \beta$

as $n \to \infty$, so there are finitely many choices for $\alpha^{-n}\beta\alpha^n$; but this

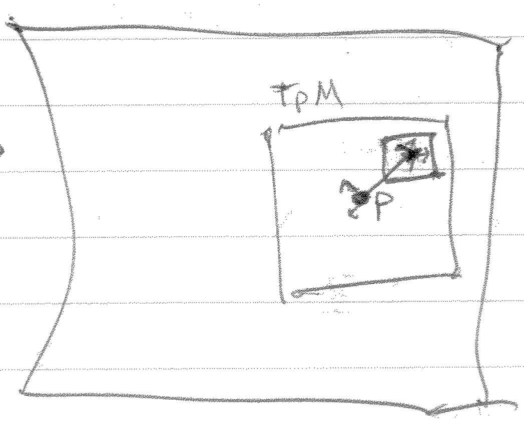
implies $\alpha^n \in Z(\langle\beta\rangle)$ = virtually cyclic, i.e. ~~α, β share same~~ $\exists N, M$: $\alpha^N = \beta^M$)

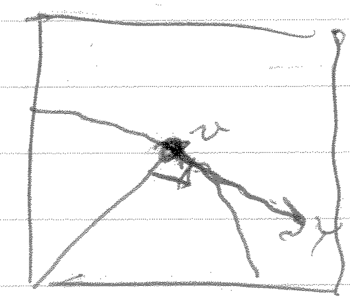
V

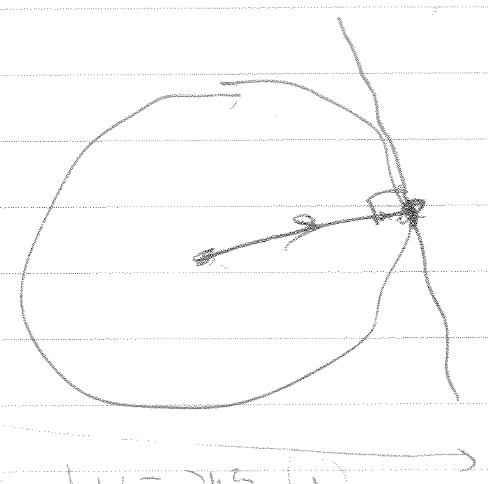
From Dynamics in Houston.

Alex Blumenthal →

You







(VI) ~~[illegible]~~ Beyond δ-hyperbolic groups

- CAT(0) groups : $G \curvearrowright X$ CAT(0)

  a slightly different generalization of $K \leq 0$

$\boxed{\text{Thm (Agol, Wise) } G \text{ hyperbolic, cubulated} \Rightarrow G \text{ virtually embeds in a RAAG}}$

$\boxed{\text{Thm (Haglund-Wise, Kahn-Markovic) } \pi_1(M_{hyp}) \text{ for } M \text{ a closed hyperbolic 3-mfld is cubulated}}$

$\Rightarrow$ Virtually Haken Thm.

- Groups from topology / geometry, e.g. $Mod(S) \curvearrowright Teich(S)$, $\curvearrowright C(S)$

- Growth, divergence, isoperimetric (Dehn) functions

  Gromov's theorem on groups of polynomial growth

  G has polynomial growth iff it is virtually nilpotent.

- ...