Ref Donaldson: Geometric Analysis Notes
Booß-Bavnbek + Bleecker: Index Theory
Chavel: Eigenvalues in Riem. Geom.
P. Li: Geom. Analysis

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Elliptic operators, Spectral Geometry, and the Index Theorem

(1)

Goal: Introduce elliptic operators and their fundamental properties, local and global, and apply to geometry.
- Weyl's law
- Index Theorem: Chern-Gauss-Bonnet, Signature, Hirzebruch-R.R.

- avoid meticulous details of analysis, but try to make the ideas cle
- avoid Fourier analysis, but it is often lurking in the background
- Many angles at which the subject can be approached.
- use more 'modern' approach first, using Functional analysis

Elliptic differential operators

Basic example $\Delta = -\left(\frac{\partial}{\partial x_1}\right)^2 - \dots - \left(\frac{\partial}{\partial x_n}\right)^2$ on $\mathbb{R}^n$

First generalization: (Scalar valued ($\mathbb{R}$) on $U \subset \mathbb{R}^n$ open)

def A scalar valued (linear) differential operator of order k is a map $P: C^\infty(U) \to C^\infty(U)$ of the form

$$Pu = \sum_{|I| \le k} a_I(x) D^I u$$

$I = (i_1, \dots, i_n)$, $D^I u = \frac{\partial^{|I|}}{(\partial x_1)^{i_1} \cdots (\partial x_n)^{i_n}} u$, $a_I(x) \in C^\infty(U)$, and not all $a_I(x) \equiv 0$ for $|I| = k$.

(2) *ex* for Δ, $a_{(0,\dots,2,\dots,0)}(x) = -1$, all else are 0.

rmk ellipticity is a condition on the highest order terms.

def The principle symbol (highest order part of the symbol) of P is the polynomial $\sigma_P(\xi) = \sum_{|I|=k} a_I(x)\, \xi_1^{i_1} \cdots \xi_n^{i_n}$.

[compare w/ F.T.] $[\sigma_{P \circ Q} = \sigma_P \circ \sigma_Q$ (chain rule), $\sigma_{P+Q} = \sigma_P + \sigma_Q]$

def P is called elliptic if $\sigma_P(\xi) \neq 0 \quad \forall x \in U, \forall \xi \in \mathbb{R}^n \setminus \{0\}$

Special case $k=2$: We can write the quadratic form σ_P as an $n \times n$ symmetric matrix, so it is diagonalizable. Then P is elliptic

$\Leftrightarrow$ the quadratic form is positive or negative definite $(\forall x)$

$\Leftrightarrow$ all eigenvalues >0 or all e.v. <0.

In this case we say P is uniformly elliptic if $\exists\, \varepsilon > 0$ s.t. $\varepsilon <$ all eigenvalues $\forall x \in U$. (or if σ uniformly bounded for $\|\xi\| = 1$)

rmk We will quickly move to the case of a compact manifold where elliptic $\Leftrightarrow$ uniformly elliptic.

③

The formal adjoint

$P^*: C^\infty(U) \to C^\infty(U)$ s.t.

uniquely characterized by

$\langle Pu, v\rangle = \langle u, P^* v\rangle \quad \forall u, v \in C_0^\infty(U)$ [actually only need one compactly supported]

$\langle u, v\rangle = \int_U u \cdot v \, dx$

Integrate by parts to move derivatives from u to v.

$\sigma_{P^*} = (-1)^k \sigma_P$, so P elliptic $\iff P^*$ is.

Vector bundles

First, consider 2 trivial vector bundles $U \times \mathbb{R}^s$, $U \times \mathbb{R}^t$.

A differential operator of order k is a map

$P: \Gamma(U \times \mathbb{R}^s) \to \Gamma(U \times \mathbb{R}^t)$ $\quad [\Gamma(U \times \mathbb{R}^s) = C^\infty(U, U \times \mathbb{R}^s)]$

of the form

$$P\alpha = \sum_{|I| \le k} a_I(x) D^I \alpha$$

Same conditions as before, but now $a_I(x) \in \Gamma(U \times \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^t))$,

that is, at each $x \in U$, have an $t \times s$ matrix.

x $d: C^\infty(M) \to \Gamma(T^*M))$

<u>Symbol</u>: $\sigma_P(\xi) = \sum_{|I|=k} a_I(x) \xi_1^{i_1} \cdots \xi_n^{i_n}$ is a homogeneous poly in ξ

with coeff. in $\Gamma(U \times \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^t))$.

④ P is called <u>elliptic</u> if $\sigma_P(\xi)$ is an isomorphism $\forall x \in U$, $\forall \xi \in \mathbb{R}^n \setminus \{0\}$,

ie, $\forall \xi \in \mathbb{R}^n \setminus \{0\}$, it is a vector bundle isomorphism.

<u>rmk</u> (i) elliptic $\Rightarrow t = s$

(ii) Sometimes, P is called elliptic if $\sigma_P(\xi)$ is injective $\forall x, \xi \neq 0$ which is enough for elliptic regularity, but not enough for our applications.

<u>rmk</u> The derivatives in P are taken with respect to the coordinates of U, and they are "exchanged" <u>linearly on the fibers.</u>

<u>def</u> $\langle \alpha, \beta \rangle_{L^2} = \int_U \langle \alpha, \beta \rangle_{\mathbb{R}^s} \, dx$ on $U \times \mathbb{R}^s$

as before, define P^* by integration by parts

$$P^* v = \sum_{|I| \leq k} (-1)^{|I|} D^I \left((a_I(x))^* v \right) \qquad \left[(a_I(x))^* = \text{conj. transpose} \right]$$

<u>Note</u>: Derivatives of $a_I(x)$ go to lower order terms, so

$$\sigma_{P^*}(\xi) = (-1)^k (\sigma_P(\xi))^*$$

So P elliptic $\Leftrightarrow$ P^* elliptic.

(5) <u>Manifolds</u> Let M be a compact, (oriented), (Riemannian) smooth n-manifold and E, F be two vector bundles over M (Sorn of rank s)

<u>def</u> $P: \Gamma(E) \to \Gamma(F)$ is called a differential operator of order k iff in local trivializations of $E + F$, P is a differential operator as before.

<u>rmk</u> The notions (except formal adjoint) are local ones. need to show independent of choice of coordinates. (product rule. chain rule on M)

<u>rmk</u> When changing coordinates of E, F, by product rule, taking a derivative of the transition functions lowers the order of derivative on the section of E, so the <u>Principle part</u> of the symbol is invariant.
[compare with jet bundles]

Therefore, ellipticity is well defined.

Glue together locally constructed formal adjoints to construct P^*:
Let $\mathcal{U}$ be an open cover. $P_u^* \beta$ defined as before for $\beta \in \Gamma_0(F)$, $u \in \mathcal{U}$.

$$P^* \beta := \sum_{u \in \mathcal{U}} P_u^* (\varphi_u \beta) \quad , \varphi_u \text{ a P.o.U sub to } \mathcal{U}.$$

$\sigma_{P^*}(\xi) = (-1)^k (\sigma_P(\xi))^*$ locally.

P elliptic $\Longleftrightarrow$ P^* elliptic

(6) ex $\Delta = -\mathrm{div}\circ\mathrm{grad} = d^*d = (d+d^*)^2 : C^\infty(M) \to C^\infty(M)$

$\Gamma^0(M,\mathbb{R})$

$\left[= -\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x_j}\right) \text{ in local coords} \right]$

$\left(\sqrt{g} = \sqrt{ev.1} \cdot \sqrt{ev.2}\right)$?

ex $\Delta = (d+d^*)^2 = dd^* + d^*d : \Omega^*(M) \to \Omega^*(M)$

ex $d+d^* : \Omega^*(M) \to \Omega^*(M)$

ex $(d+d^*)\big|_{\Omega^{even}} : \Omega^{even} \to \Omega^{odd}$

ex $(d+d^*)\big|_{\Omega^+} : \Omega^+ \to \Omega^-$ [self dual + anti-self dual forms w.r.t. intersection form + eigenspaces of $(i)^m *$]

ex $\bar\partial + \bar\partial^*$: Dolbeault cx ⟲ $[\Omega^{0,*} \otimes E]$

ex $(\bar\partial + \bar\partial^*)$: even deg Dolbeault $\to$ odd deg Dolbeault

⑦ A more global definition of σ_P:

Note that in Principle symbol $\frac{\partial}{\partial x_i}$ transform like tangent vectors,

and partial derivatives commute, so ∂^I transform like symmetric powers of $T(T^*M)$. [alternatively, a covariant symmet. tensor]

And $a_I(x) \in \Gamma(\mathrm{Hom}_{\mathbb{R}}(E), \mathrm{Hom}(F))$, so

$\sigma_P \in \Gamma(\mathrm{Hom}(E,F) \otimes S^k TM)$

globally, Think of $\xi = (\xi_1, \dots, \xi_n)$ as a covector $\xi \in T^*M$.

At $x \in M$, $\sigma_P(\xi)(\alpha) = P(f^k \cdot \alpha)$ where $df = \xi$, $f(x)=0$, $[f] = \xi$.

[to compare, note $f(x)=0 \Rightarrow$ only non-zero terms after using ~~product~~ rule are when k derivatives fall on f, so only get highest order terms, with no derivatives on α]

ex $\sigma_{(d+d^*)}(\xi)(\alpha) = \xi \wedge \alpha \underset{?}{\pm} \xi \lrcorner \alpha$

ex $\sigma_\Delta(\xi) = -\|\xi\|_g^2 \, Id_{\Omega^*(M)}$

rmk $\sigma_{P \circ Q} = \sigma_P \circ \sigma_Q$ [chain rule]

$\sigma_{P+Q} = \sigma_P + \sigma_Q$ [trivial]