The Fundamental Theorem of Calculus for Line Integrals

What to know:

1. Be able to state the FTC for conservative vector fields.

2. Know that line integrals of **conservative** vector fields only depend on initial and terminal point of the path, and are 0 along closed paths.

3. Be able to determine if a set is simply connected by looking at a picture.

4. Be able to check if a vector field defined on a subset of $\mathbb{R}^2$ is conservative.

5. Be able to find a potential function for a conservative vector field and use it to compute line integrals.

Recall from Math 125 that the Fundamental Theorem of Calculus was a tool that related the integral of the derivative of a function over an interval with its values at the endpoints:

$$\int_a^b f'(x)dx = f(b) - f(a) \tag{1}$$

Our goal is to generalize this for line integrals of vector fields. Suppose $\vec{F}(x,y) = \nabla f(x,y)$ is a conservative vector field on a domain D, and a curve $c(t) = \langle x(t), y(t)\rangle$, for $t \in [a,b]$, contained in D. Then, we may write

$$\begin{aligned}\int_c \vec{F} \cdot d\vec{r} = &\int_c \nabla f \cdot d\vec{r} \\ =&\int_a^b f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t)dt \\ =&\int_a^b \frac{d}{dt}(f(\vec{r}(t))dt \\ =&f(c(b)) - f(c(a)),\end{aligned}$$

using the chain rule and the FTC.

This gives us our theorem:

Theorem 1. *Let $c(t) = \langle x(t), y(t)\rangle$, $t \in [a,b]$ be a piecewise smooth curve in a domain D and $\vec{F} = \nabla f$ a conservative vector field on D. Then*

$$\int_c \vec{F} \cdot d\vec{r} = f(c(b)) - f(c(a)). \tag{2}$$

The same theorem holds for curves in $\mathbb{R}^3$ as well.

Example 1. *Find the work produced by the gravitational force*

$$\vec{F}(x,y,z) = -\frac{mMG}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

between an object of mass M at the origin and an object of mass m at (x,y,z), while the latter is moving along the path $c(t) = \langle \sin(t), \cos(t), t/2\pi\rangle$, $t \in [0,2\pi]$.

Solution. Recall that the gravitational vector field is conservative and $f(x, y, z) = \dfrac{mMG}{(x^2 + y^2 + z^2)^{1/2}}$ is a potential function for it. So, by FTC,

$$\begin{aligned}\int_c \vec{F} \cdot d\vec{r} =& f(c(b)) - f(c(a)) \\ =& f(0, 1, 1) - f(0, 1, 0) \\ =& \frac{mMG}{\sqrt{2}} - \frac{mMG}{1}\end{aligned}$$

□

Remarks: Look at the right hand side of (2): c doesn't appear, only its initial and terminal points. Therefore, **for conservative vector fields**, the line integral does **not** depend on the path, only on its initial and terminal point. Stating this more formally, if $\vec{F}$ is a conservative vector field in a domain D and c_1, c_2 are two curves in D so that $c_1(a) = c_2(a)$ and $c_1(b) = c_2(b)$ then

$$\int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r}.$$

In addition, if the path is closed (that is, $c(a) = c(b)$), then the integral of a **conservative** vector field is 0! This is the justification for calling a vector field conservative: the energy is conserved along closed paths: that is, whatever energy the force gives to an object moving inside a closed path, it takes it back; it can't be providing energy all the time! In the example of the gravitational vector field, think of when you jump: gravity gives you potential energy while you are ascending, but you lose it while descending. In contrast to that, friction is not conservative: it only removes energy from a moving object without ever giving any of it back.

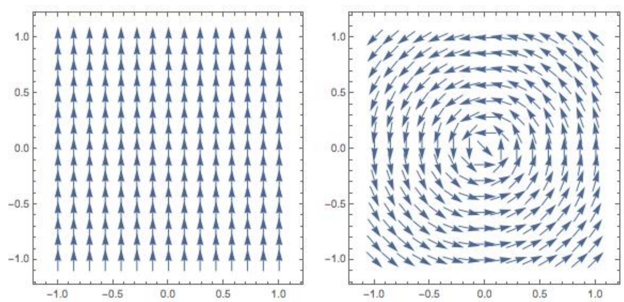


Figure 1: Which of the two vector fields is more likely to be conservative?

Why is the property of being conservative useful? Because if we have a conservative field and want to compute the line integral along a path we can make our lives easier by using one of the following two ideas (mainly the second one):

1. integrate over a very simple path connecting the endpoints of the path, such as a line segment; it doesn't matter what we choose, we should find the same answer. For instance, in the previous example you could also integrate $\vec{F}$ over the line segment connecting $(0, 1, 0)$ and $(0, 1, 1)$.

2. more importantly, find a potential function and use the FTC.

How can we tell if a vector field is conservative?

In your first calculus courses, you might remember that once we had a continuous function g, we could always find a function h so that $h' = g$, called an antiderivative, and we did that by integrating, i.e. setting

$$h(x) = \int_a^x g(t)dt.$$

Question: Is it possible to do something analogous for two or three dimensions, that is, once we have a continuous or differentiable vector field, can we "integrate" it and find a potential function? The answer is **sometimes**.

Suppose we have a vector field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$$

defined on $D \subset \mathbb{R}^2$, where p and Q are continuously differentiable. Then, if we assume that $f(x, y)$ is a potential function, we'd have

$$\begin{aligned} P(x, y) =& f_x(x, y) \\ Q(x, y) =& f_y(x, y) \end{aligned}$$

Differentiating the first equation with respect to y and the second with respect to x, we find

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y) \text{ for all } (x, y) \in D,$$

by Clairaut's theorem. So we have the following theorem:

Theorem 2. *If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$, defined on $D \subset \mathbb{R}^2$, where p and Q are continuously differentiable, is a **conservative** vector field, then*

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y) \textit{ for all } (x, y) \in D$$

Remark: Here it's important that we are doing this in 2 dimensions, it looks different in 3!

So, if we are given a vector field on a domain D and $\frac{\partial P}{\partial y}(x, y) \neq \frac{\partial Q}{\partial x}(x, y)$ even at one point in D, $\vec{F}$ is not conservative! That is, this theorem can tell that a vector field is **not** conservative, if $\frac{\partial P}{\partial y}(x, y) \neq \frac{\partial Q}{\partial x}(x, y)$ somewhere, but it doesn't say much about whether it is.

Question: Can we say that the converse is true? That is, if we know that

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y) \text{ for all } (x, y) \in D$$

does this tell us that the vector field is conservative? Do the following exercise:

Exercise 1. *Let $\vec{F}(x, y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle = \langle P(x, y), Q(x, y)\rangle$, defined on $\mathbb{R}^2 \backslash \{(0, 0)\}$ (the plane without the origin).*

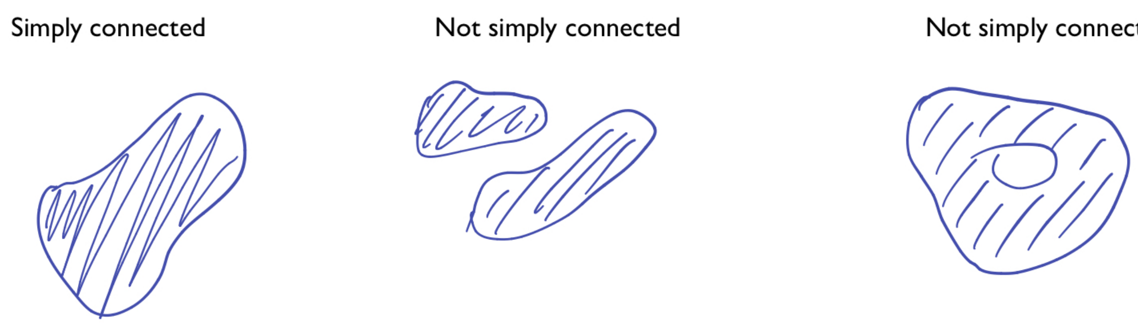
1. *Compute $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ on D.*

2. *Compute $\int_c \vec{F} \cdot d\vec{r}$, where $c(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$ (the unit circle).*

If you did the previous exercise, you'd find that the integral over a closed path is not zero. The reason why in this case $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ on D does **not** imply that $\vec{F}$ has a potential function is that the domain has a hole in the middle.

Definition 1. *(A bit informal) A domain $D \subset \mathbb{R}^2$ that consists of one piece and has no holes is called **simply connected**.*

Remark: This definition doesn't work in dimensions ≥ 3: it is a bit more complicated then.



We have the following:

Theorem 3. *If $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, defined on a **simply connected domain** $D \subset \mathbb{R}^2$, where P and Q are continuously differentiable and*

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y) \text{ for all } (x,y) \in D$$

then there exists a potential function f for $\vec{F}$ on D, that is, $\vec{F} = \nabla f$ on D.

Remark: Again, this theorem holds as is in 2 dimensions only!

How do we find a potential function once we know it exists?

Once we are given a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ in $D \subset \mathbb{R}^2$, here are the steps we take:

1. Check that $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ for all $(x,y) \in D$ and D is simply connected, to make sure the potential function exists.
2. Integrate P with respect to x to find $f(x,y) = \int P(x,y)dx + g(y)$.
3. Differentiate the result with respect to y, set $\frac{\partial f}{\partial y} = Q$.
4. Integrate with respect to y to determine g.

Let's see how this works in an example:

Example 2. *Let $\vec{F}(x,y) = \langle 2xy, x^2 + 6y^2\rangle$ on $\mathbb{R}^2$. Determine if $\vec{F}$ is conservative and find a potential function, if it is.*

Solution. We set $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ and find that

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(2xy) = 2x \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}(x^2+6y^2) = 2x\end{aligned}$$

so, since $\mathbb{R}^2$ is simply connected, $\vec{F}$ is conservative. Let's find a potential function, call it f. We need $\vec{F} = \nabla f$, so

$$f_x = 2xy \tag{3}$$

and

$$f_y = x^2 + 6y^2. \tag{4}$$

Then integrating the first one with respect to x, we find

$$f(x,y) = x^2y + \phi(y).$$

This is saying that the constant of integration with respect to x doesn't have a reason to not depend on y! Therefore, we have to think of it as a function of y.

To make use of (4), we differentiate with respect to y:

$$f_y = x^2 + \phi'(y). \tag{5}$$

So, using (4),

$$x^2 + \phi'(y) = x^2 + 6y^2$$

and we may finally integrate with respect to y and find

$$\phi(y) = 2y^3 + c,$$

and therefore

$$f(x,y) = x^2y + 2y^3 + c.$$

□

Finding potential functions in $\mathbb{R}^3$

The above discussion doesn't give information about determining whether a vector field in $\mathbb{R}^3$ is conservative. We will see such a criterion in the sections to come.

Here's how we find a potential function for a vector field on a subset of $\mathbb{R}^3$ once it is given that it is conservative.

Example 3. *It is known that $\vec{F}(x,y,z) = \langle 2x, 6zy^2, 2y^3+2\rangle$ is conservative. Find a potential function f for it.*

Solution. Set

$$\begin{aligned}P(x,y,z) &= 2x \quad (6)\\ Q(x,y,z) &= 6zy^2 \quad (7)\\ R(x,y,z) &= 2y^3 + 2. \quad (8)\end{aligned}$$

Then $f_x = P \Rightarrow f_x = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z)$, for some $g(y, z)$. Then $f_y = g_y$, so

$$g_y = Q = 6zy^2.$$

This gives

$$g(y, z) = 2zy^3 + h(z).$$

So

$$f(x, y, z) = x^2 + 2zy^3 + h(z) \Rightarrow f_z = 2y^3 + h'(z).$$

Finally, $f_z = R = 2y^3 + 2$ means $h'(z) = 2 \Rightarrow h(z) = 2z + c$. Therefore,

$$f(x, y, z) = x^2 + 2zy^3 + 2z + c.$$

□

An optional remark about Exercise 1

In Exercise 1, you should have found that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on $D = \mathbb{R}^2 \backslash \{(0, 0)\}$. As we saw, we can't find a potential function for $\vec{F}$ on D, since this would have to imply that the line integral over the unit circle is 0. However, we could still restrict our interest in any simply connected subset of $\mathbb{R}^2 \backslash \{(0, 0)\}$ and find a potential function there (for example, you could take a disk of radius 1 centered at (2,0)).

You can follow the procdure described before for computing potential functions, and find that in any such domain a potential function for $\vec{F}$ is $f(x, y) = \arctan(\frac{y}{x})$, which looks more familiar in polar coordinates: $f(\rho, \theta) = \theta!$. This function can be defined to be differentiable in any simply connected subset of D, but not everywhere on D: after completing a full rotation, the angle θ has jumped by 2π and therefore can't be continuous!.