

- Office Hours: Tue & Thurs, 6-7 pm, Zoom (see Zoom # on Brightspace)
(please email me in advance if you would like to meet)
another day+time
- Homework
 - Online - to be done online in MyLab Math
 - Handwritten - to be submitted via Gradescope

↓
There will be 3 online + 3 written (short) homework assignments each week (due 11:59 pm of the coming Monday)

↓
you will complete 36 online + 36 written homeworks through the entire term, each worth of 20 points
(at the end of the term, 3 lowest scores in each category will be dropped)

- Lectures → live-streamed + recorded (and posted on Brightspace)

Today: Using document camera in classroom

Thursday: Using Zoom (or Webex in case of issues with Zoom) + doc. camera

After trying those two strategies, we shall see which one works better for you and stick to it till the end of the semester.

! Don't worry about IT glitches: if something goes wrong, I will re-record the video AND I will post the notes of each class both on Brightspace as well as personal webpage.

Lecture #1

§1.1 Systems of linear equations

Def: A linear equation in the variables x_1, \dots, x_n is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

Here, b and the coefficients a_1, \dots, a_n are usually given numbers

Examples: 1) $x - 5y = 2$ (as it is equivalent to $x_1 - 5x_2 = 2$)

Note: As we shall work with linear equations in 4, 5, ... variables we prefer to use (x_1, x_2) instead of (x, y) .

Here: $a_1 = 1, a_2 = -5, b = 2$

2) $x_1 + 3x_2 + 10 = 0$ (as it is equivalent to $x_1 + 3x_2 = -10$)

here: $a_1 = 1, a_2 = 3, b = -10$.

3) $x_1 + x_2 - 10x_3 = \pi \cdot x_1 + 1000$ (as it is equivalent to $(1-\pi)x_1 + x_2 - 10x_3 = 1000$)

here: $a_1 = 1-\pi, a_2 = 1, a_3 = -10, b = 1000$

Non-examples: 1) $x_1 + 2x_2 = x_3^2$

2) $x_1 + \sin(x_2) - 5x_3 = 2$

← In this course, we should never worry about such complicated equations.

Def: A linear system (a.k.a. system of linear equations) is a collection of linear equations involving the same variables, say x_1, x_2, \dots, x_n .

Example: $x_1 - 3x_2 + x_3 = 3$

← here: $a_1 = 1, a_2 = -3, a_3 = 1, b = 3$

$x_1 - x_3 = 1$

← here: $a_1 = 1, a_2 = 0, a_3 = -1, b = 1$

Note: 1) It will be convenient for the future discussions to write 2nd equation in the above form, leaving out a blank spot for $0 \cdot x_2$.

2) You can also combine all equations involved by drawing { on the left of those.

Lecture #1

Let's start from a very simple highschool-level exercise:

Exercise #1: Find all solutions of the system

$$\begin{aligned}x_1 + x_2 &= 2 \\ 2x_1 - x_2 &= 1\end{aligned}$$

Solution(s)

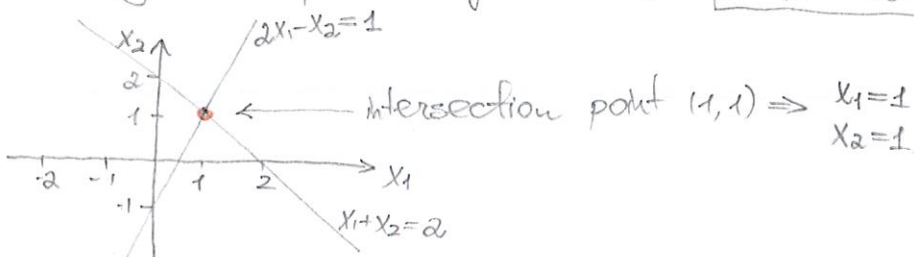
There are three different ways to proceed

Solution 1 (via substitution): $x_1 = 2 - x_2$ (from 1st eq-n) $\xrightarrow{\text{plug into 2nd eq-n}}$ $\underbrace{2(2 - x_2) - x_2 = 1}_{3 - 3x_2 = 0}$

$$\Rightarrow \boxed{x_2 = 1 \Rightarrow x_1 = 1}$$

Solution 2 (via elimination): Adding two equations, find $3x_1 = 3 \Rightarrow \boxed{x_1 = 1 \Rightarrow x_2 = 1}$

Solution 3 (by graphing):



Note: From the last (geometric) approach it becomes clear that given two ^{linear} equations in two variables, they determine two lines, which are

- 1) intersect at 1 point, or
- 2) parallel, or
- 3) coincide.

And this is a very general feature!

CLAIM: Any linear system has

1) no solutions

OR

2) exactly one solution

OR

3) infinitely many solutions

← A solution of the linear system is an n -tuple (s_1, s_2, \dots, s_n) of numbers such that each equation holds as you substitute s_1, \dots, s_n for x_1, \dots, x_n .

Q: Which of those 3 approaches works better for linear systems in > 2 variables?

A:

- Geometric approach is hard to visualize (though for 3 variables, it boils down to intersection of planes in space)
- Substitution is ok, but becomes quite cumbersome.

So: we shall use the "elimination approach", but using some new notation. ③

Lecture #1

Exercise #2: Solve (i.e. find all solutions) the following linear system:

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 4 \\ x_2 - x_3 = 2 \end{cases}$$

As we shall be eliminating the variables one-by-one, all the essential information about the original linear system can be encoded by the 3×4 matrix (which is a rectangular array with 3 rows and 4 columns)

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & 2 \end{array} \right) \leftarrow \text{called "augmented matrix" of the above system.}$$

Note: 1) In textbook they use $[\dots]$ and do not draw vertical line.
2) Erasing last column, get the coefficient matrix of the system.

Solution

• To eliminate variable x_1 , we need to multiply 1st eq-n by 2 and subtract from the 2nd equation, which on the level of matrices looks as follows:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & 2 \end{array} \right) \xrightarrow{R_2 - 2 \cdot R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array} \right)$$

• Next, we switch the order of the 2nd and 3rd equations, which on the level of matrices looks as follows:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & 1 & 2 \end{array} \right)$$

• Next, we eliminate x_2 from the 3rd equation, which is depicted via:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & 1 & 2 \end{array} \right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \end{array} \right)$$

Now, we are almost done, as we ended up with an equivalent linear system $\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_2 - x_3 = 2 \\ 4x_3 = -4 \end{cases}$

• From the 3rd eq-n get $x_3 = -1$
 \rightarrow plug $x_3 = -1$ into 2nd eq-n to get $x_2 = 1$
 \rightarrow plug both in 1st eq-n to get $x_1 = 3$

So: The only solution is $(3, 1, -1)$, equivalently $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

Lecture #1

The above solution illustrates how "elimination procedure" works on the level of the corresponding augmented matrices. To be precise, one is applying so-called elementary row operations:

- 1) Replace a row by a sum of itself and a multiple of another row
- 2) Interchange two rows
- 3) Multiply all entries of a row by a nonzero constant

Note: All these operations are reversible!

As a result, we get

one is obtained from another via elementary row operations

CLAIM: If the augmented matrices of two linear systems are row equivalent then the two systems have the same solution set.

Def: A linear system is called consistent if admits at least one solution.
A linear system is called inconsistent if it has no solution.

Exercise #3: Determine the value(s) of g such that the matrix $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & g & 10 \end{array} \right)$ is the augmented matrix of a consistent linear system

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & g & 10 \end{array} \right) \xrightarrow{R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & g-8 & -2 \end{array} \right)$$

If $g \neq 8 \Rightarrow$ can recover x_2 uniquely ($x_2 = -\frac{2}{g-8}$) \Rightarrow recover x_1

If $g = 8 \Rightarrow$ the 2nd equation reads $0 = -2 \Rightarrow$ does not hold.

So: $g \neq 8$.

Exercise #4: Construct two different augmented matrices for linear systems with solutions $x_1 = 1$, $x_2 = 5$, $x_3 = -10$.

E.g. the simplest one is $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -10 \end{array} \right)$

To obtain some other, apply elementary row operations, e.g. $R_2 - R_1$ gives:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -10 \end{array} \right)$$

Lecture #1

§1.2 Row Reduction and Echelon Form

Let's start from a couple of definitions, which apply to any matrix (not necessarily viewed as the augmented matrix)

Def: A rectangular matrix is in echelon form if:

- 1) All nonzero rows are above any rows of all zeros
- 2) Each leading entry of a row is in a column to the right of the (i.e. leftmost nonzero entry) leading entry of the row above it.
- 3) All entries in a column below a leading entry are zeros

Moreover, if it satisfies 4)-5) below, then a matrix is in reduced echelon form

- 4) The leading entry in each nonzero row is 1
- 5) Each leading 1 is the only nonzero entry in its column.

Def: A pivot is the same as the leading entry of a row, i.e. the leftmost nonzero entry of a row.

Example: The matrix $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & 4 \end{array} \right)$ we ended with in the solution of Exercise 2 is in echelon BUT not reduced echelon form.

However, performing elementary row operations to it:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \end{array} \right) \xrightarrow{\frac{1}{4}R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right) \xrightarrow{\begin{array}{l} R_2+R_3 \\ R_1-R_3 \end{array}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right)$$

we end up with $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right)$ which is in the reduced echelon form.

CLAIM: Each matrix is row equivalent to one and only one reduced echelon matrix