Lecture #2

- <u>Last time</u>: 1) Linear eq-n, linear system, solution of a linear system

    2) Elimination of variables to solve a linear system via <u>elementary row operations</u>, applied to the augmented matrix.

    ↓

    (Recall them!)

    3) Notions of <u>echelon</u> AND <u>reduced echelon</u> forms

- <u>Warming up</u>

<u>Ex1</u>: For which values of numbers c, d the linear system

$(*)$ $\begin{cases} x_1 + c\,x_2 = 3 \\ 2x_1 - 4x_2 = d \end{cases}$

1) has exactly one solution

2) has infinitely many solutions

3) is <u>inconsistent</u> (i.e. has no solutions)

$\begin{pmatrix} 1 & c & | & 3 \\ 2 & -4 & | & d \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 2 \cdot R_1} \begin{pmatrix} 1 & c & | & 3 \\ 0 & -4-2c & | & d-6 \end{pmatrix}$ ⤳ corresponds to system $\begin{cases} x_1 + c x_2 = 3 \\ (-4-2c)x_2 = d-6 \end{cases}$

<u>Case 1</u>: $-4-2c \neq 0$ (equivalently, $c \neq -2$)

Then, x_2 is uniquely recovered: $x_2 = \frac{d-6}{-4-2c}$. Plugging this value of x_2 into the 1st equation uniquely recovers x_1. <u>Thus</u>: there is exactly 1 solution.

<u>Case 2</u>: $-4-2c = 0$ (i.e. $c = -2$)

I) if $d-6 \neq 0$ (i.e. $d \neq 6$), then there is no x_2 such that $0 \cdot x_2 = d-6$

⟹ <u>no solutions</u>

II) if $d-6 = 0$ (i.e. $d = 6$), then 2nd equation reads $0 \cdot x_2 = 0$ and any value of x_2 works. For any x_2, the 1st equation recovers x_1 uniquely. So: there are infinitely many solutions.

<u>Conclusion</u>:
1) $(*)$ has exactly <u>one</u> solution iff $c \neq -2$
2) $(*)$ has <u>infinitely</u> many solutions iff $c = -2$ & $d = 6$
3) $(*)$ has <u>no</u> solutions iff $c = -2$ AND $d \neq 6$

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Before we proceed to the next example, let's recap 2 definitions:

Def: A pivot position in a matrix A is a location that corresponds to a leading entry 1 in the reduced echelon form.

Def: A pivot column is a column of A that contains a pivot position.

! Note: Pivot position and pivot column are already seen from an echelon form

e.g. in $\begin{pmatrix} \boxed{2} & 3 & -5 & 7 & 10 \\ 0 & \boxed{5} & 0 & 6 & 12 \\ 0 & 0 & 0 & \boxed{-1} & 3 \end{pmatrix}$ the boxed positions are pivot positions

pivot columns

Ex 2: Apply elementary row operations to transform

$$\begin{pmatrix} 0 & 1 & 2 & -3 & 1 & 5 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 1 & 2 & 7 & -2 & 0 & 17 \end{pmatrix}$$

first into an echelon form and then into the reduced echelon form.

$\begin{pmatrix} 0 & 1 & 2 & -3 & 1 & 5 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 1 & 2 & 7 & -2 & 0 & 17 \end{pmatrix}$ $\xrightarrow{R_1 \leftrightarrow R_3}$ $\begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 0 & 1 & 2 & -3 & 1 & 5 \end{pmatrix}$ $\xrightarrow{R_2 \mapsto R_2 - 2R_1}$

$\xrightarrow{}$ $\begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & -3 & -6 & 9 & 0 & -18 \\ 0 & 1 & 2 & -3 & 1 & 5 \end{pmatrix}$ $\xrightarrow{R_2 \leftrightarrow R_3}$ $\begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & 1 & 2 & -3 & 1 & 5 \\ 0 & -3 & -6 & 9 & 0 & -18 \end{pmatrix}$ $\xrightarrow{R_3 \mapsto R_3 + 3R_1}$

$\xrightarrow{}$ pivots $\begin{pmatrix} \boxed{1} & 2 & 7 & -2 & 0 & 17 \\ 0 & \boxed{1} & 2 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 & \boxed{3} & -3 \end{pmatrix}$ $\xrightarrow{R_3 \mapsto \frac{1}{3}R_3}$ $\begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & 1 & 2 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$ $\xrightarrow{R_2 \mapsto R_2 - R_3}$

already in echelon form
BUT <u>not</u> in reduced echelon

$\xrightarrow{}$ $\begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & 1 & 2 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$ $\xrightarrow{R_1 \mapsto R_1 - 2R_2}$ $\begin{pmatrix} \boxed{1} & 0 & 3 & 4 & 0 & 5 \\ 0 & \boxed{1} & 2 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{pmatrix}$

reduced echelon form!

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Let us now relate these row operations back to solving linear systems.

Ex 3 : Solve the following linear system

$$\begin{cases} x_2 + 2x_3 - 3x_4 + x_5 = 5 \\ 2x_1 + x_2 + 8x_3 + 5x_4 = 16 \\ x_1 + 2x_2 + 7x_3 - 2x_4 = 17 \end{cases}$$

We start by writing down the corresponding augmented matrix :

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & -3 & 1 & 5 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 1 & 2 & 7 & -2 & 0 & 17 \end{array} \right)$$

As this is exactly the matrix from Ex 2, we know by now that it is row equivalent to the following reduced echelon matrix:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 4 & 0 & 5 \\ 0 & 1 & 2 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right)$$

Let us now write down the corresponding linear system:

$$\begin{cases} x_1 + 3x_3 + 4x_4 = 5 \\ x_2 + 2x_3 - 3x_4 = 6 \\ x_5 = -1 \end{cases}$$

The 3rd equation determines x_5 uniquely : $\boxed{x_5 = -1}$

But for any values of x_3 and x_4, the values of x_2 and x_1 are uniquely determined by 2nd & 1st equations :

$$x_2 = 6 - 2x_3 + 3x_4 \quad \underline{AND} \quad x_1 = 5 - 3x_3 - 4x_4.$$

NOTE : x_3, x_4 - free , i.e. can take any values!

Answer : $\begin{cases} x_1 = 5 - 3x_3 - 4x_4 \\ x_2 = 6 - 2x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = -1 \end{cases}$

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In the above examples, the variables x_1, x_2, x_5, corresponding to pivot columns are called "basic variables", while the other variables x_3, x_4 are called "free variables"

Rmrk: a) The "free variables" can take any values, while "basic variables" will be uniquely determined by them.

   b) Returning to the system

$$\begin{cases} x_1 + 3x_3 + 4x_4 = 5 \\ x_2 + 2x_3 - 3x_4 = 6 \\ x_5 = -1 \end{cases}$$

we could also let x_2, x_3 to take any values, then the 2nd equation would uniquely determine x_4, and then the 1st equation would uniquely determine x_1.

   ! However, it is customary to make "free variables" to take any values.

Rmk: While reducing a matrix to an echelon form we move from top rows to bottom rows, when transforming echelon to a reduced echelon form, we move backwards: from bottom rows to top rows (note: this is equivalent to substitution algorithm)

Q: What type of solutions does the system of Ex3 have?
(Answer: infinitely many)

Using the above argument, one arrives at the following result:

CLAIM: a) A linear system is consistent iff the right most column of the augmented matrix is NOT a pivot column, i.e. there is no row $(0 \dots 0 \mid b \neq 0)$ in its echelon form

   b) If a linear system is consistent, then:
   I) it has a unique solution iff there are no free variables
   II) it has infinitely many solutions when there is at least one free variable.

↰ Present/Discuss why this is so...
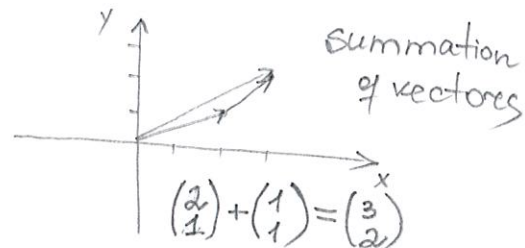
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Let us now move to

§1.3 Vector equations

Def: A matrix with only one column is called a <u>column vector</u> (or just a <u>vector</u>)

Examples: $\begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \mathbb{R}^2$

↑ 2-dimensional real plane
symbol "$\in$" means "belongs to".

summation of vectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Def: For any positive integer n, $\mathbb{R}^n$ denotes the collection of all $n \times 1$ column vectors.

There are two basic operations one can do with vectors:

1) <u>Addition</u>, e.g. $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix}$

↖ Both are defined coordinate-wise.

2) <u>Scalar Multiplication</u>, e.g. $d \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \cdot a \\ d \cdot b \\ d \cdot c \end{pmatrix}$ ↙

<u>Natural Properties</u> (assume $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ while $a, b \in \mathbb{R}$):

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$\vec{u} + \vec{o} = \vec{u} = \vec{o} + \vec{u}$ ← here: $\vec{o} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ — so-called <u>zero vector</u>

$a(\vec{u} + \vec{v}) = a \cdot \vec{u} + a \cdot \vec{v}$

$(a + b)\vec{u} = a\vec{u} + b\vec{u}$

$a(b\vec{u}) = (ab)\vec{u}$

$1 \cdot \vec{u} = \vec{u}$

we use arrows above to indicate that those are vectors

Def: Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$ and scalars $c_1, \underline{\quad} c_k \in \mathbb{R}$, we call $\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \underline{\quad} + c_k \vec{v}_k$ a <u>linear combination</u> of $\vec{v}_1, \underline{\quad} \vec{v}_k$ (with <u>weights</u> $c_1, \underline{\quad} c_k$).

Def: Given $\vec{v}_1, \vec{v}_2, \underline{\quad} \vec{v}_k \in \mathbb{R}^n$, the <u>span of $\vec{v}_1, \underline{\quad} \vec{v}_k$</u>, denoted Span$\{\vec{v}_1, \underline{\quad} \vec{v}_k\}$ is a subset of $\mathbb{R}^n$ consisting of all linear combinations of $\vec{v}_1, \underline{\quad} \vec{v}_k$ (i.e. a subset of all $\vec{w} \in \mathbb{R}^n$ which can be written as $\vec{w} = c_1 \vec{v}_1 + \underline{\quad} + c_k \vec{v}_k$)

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Q: Describe the span of

a) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

b) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$

c) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

Let us conclude by relating vector equation

$$\boxed{x_1 \cdot \vec{v_1} + x_2 \cdot \vec{v_2} + \dots + x_k \cdot \vec{v_k} = \vec{w}}$$

to linear systems — whose augmented matrix has columns $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}, \vec{w}$.

Ex 4: Let $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix}$.

Determine if $\vec{w}$ belongs to Span $\{\vec{v_1}, \vec{v_2}\}$ or not.

This boils down to the question if there are scalars x_1, x_2 such that

$$x_1 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix}, \text{ equiv. solving } \begin{cases} x_1 - 2x_2 = -4 \\ 3x_1 + x_2 = 9 \\ -2x_1 + 4x_2 = 8 \end{cases}$$

Augmented matrix: $\begin{pmatrix} 1 & -2 & | & -4 \\ 3 & 1 & | & 9 \\ -2 & 4 & | & 8 \end{pmatrix} \xrightarrow[\substack{R_2 \mapsto R_2 - 3R_1 \\ R_3 \mapsto R_3 + 2R_1}]{} \begin{pmatrix} 1 & -2 & | & -4 \\ 0 & 7 & | & 21 \\ 0 & 0 & | & 0 \end{pmatrix}$

The latter encodes the linear system $\begin{cases} x_1 - 2x_2 = -4 \\ 7x_2 = 21 \\ 0 = 0 \end{cases} \implies \begin{array}{c} x_2 = 3 \\ \Downarrow \\ x_1 = 2 \end{array}$

Answer: Yes ($\vec{w} = 2\vec{v_1} + 3\vec{v_2}$)

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