# Lecture #3

09/01/2020

- Last time → Vectors, $\mathbb{R}^n$
  ↳ Linear combinations, Span of a collection of vectors

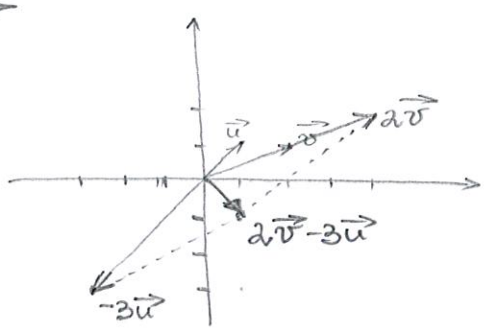
Recall: 1) Addition and scalar multiplication are done coordinate-wise
Warning: NOT TO CONFUSE with dot or cross products!

2) All vectors in this course are column vectors, i.e. we write them as a single column, which is NOT the same as a single row.
E.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \end{bmatrix}$

3) Given a vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ we can identify it with a point in the plane

— // — $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ — // — a point in the space.

Ex1: Given $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, visualize vectors $2\vec{v}$, $-3\vec{u}$, and $2\vec{v} - 3\vec{u}$.




$$\left.\begin{aligned} 2\vec{v} &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ -3\vec{u} &= \begin{pmatrix} -3 \\ -3 \end{pmatrix} \end{aligned}\right\} \Rightarrow 2\vec{v} - 3\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Note: All those vectors from above (i.e. $2\vec{v}, -3\vec{u}, 2\vec{v}-3\vec{u}$) are clearly linear combinations of $\vec{u}, \vec{v}$ (i.e. belong to Span$\{\vec{u}, \vec{v}\}$).

Q: What about $\vec{0}$?

$\vec{0} \in$ Span$\{\vec{u}, \vec{v}\}$ as $\vec{0} = 0 \cdot \vec{u} + 0 \cdot \vec{v}$

! Warning: Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, the notation $\{\vec{v}_1, \ldots, \vec{v}_k\}$ just denotes this collection, while Span$\{\vec{v}_1, \ldots, \vec{v}_k\}$ denotes their span.

For example: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ BUT $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in$ Span$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

Ex2: Given two nonzero vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$, describe what Span$\{\vec{v}_1, \vec{v}_2\}$ looks like.

If $\vec{v}_1$ and $\vec{v}_2$ are proportional to each other, then Span$\{\vec{v}_1, \vec{v}_2\}$ is a line passing through the origin.
Otherwise, Span$\{\vec{v}_1, \vec{v}_2\}$ is a plane passing through the origin.

The next exercise will allow us to relate vector equations

$$x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + \_\_\_ + x_k \cdot \vec{v}_k = \vec{w}$$

to linear systems discussed in Lecture 1.

Ex 3: Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix}$

Determine if $\underbrace{\vec{w} \text{ belongs to Span}\{\vec{v}_1, \vec{v}_2\}}$.

another way to phrase: is $\vec{w}$ a linear combination of $\vec{v}_1, \vec{v}_2$.

▶ Using the definition of scalar multiplication and vector addition, this boils down to the question if there are scalars $x_1$, $x_2$ such that

$$x_1 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix} \text{ or equivalently } \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix}.$$

Hence, we need to verify if the linear system $\begin{cases} x_1 - 2x_2 = -4 \\ 3x_1 + x_2 = 9 \\ -2x_1 + 4x_2 = 8 \end{cases}$ is consistent or not.

Augmented matrix:

$$\left(\begin{array}{cc|c} 1 & -2 & -4 \\ 3 & 1 & 9 \\ -2 & 4 & 8 \end{array}\right) \underset{R_3+2R_1}{\xrightarrow{R_2-3R_1}} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 7 & 21 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{7}R_2} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\downarrow R_1 \mapsto R_1 + 2R_2$$

$$\text{encodes linear system } \begin{cases} x_1 = 2 \\ x_2 = 3 \\ 0 = 0 \end{cases} \leftarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

Thus, there is a unique solution: $x_1 = 2$, $x_2 = 3$.

Answer: Yes! (moreover, $\vec{w} = 2\vec{v}_1 + 3\vec{v}_2$)

**!** Note that we reduced the original question to the consistency of the linear system, whose augmented matrix is formed by placing column vectors $\vec{v}_1$ $\vec{v}_2$ $\vec{w}$ one after another.

CONCLUSION: A vector equation $x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + \_\_ + x_k \cdot \vec{a}_k = \vec{b}$ has the same solution set as the linear system whose augmented matrix is $\left(\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_k & \vec{b} \end{array}\right)$

In particular, $\vec{b}$ is a linear combination of $\vec{a}_1, \_ \vec{a}_k$ iff the corresponding linear system is consistent.

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Ex 4: Consider vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$.

For which values of parameter $g$, can $\vec{w} = \begin{pmatrix} 2 \\ 1 \\ g \end{pmatrix}$ be written as a linear combination of $\vec{v}_1, \vec{v}_2$.

▶ Boils down to consistency of the linear system with augmented matrix $\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & g \end{array}\right)$. Performing elementary row operations, get:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & g \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & -2 & g-6 \end{array}\right) \xrightarrow{R_2 \mapsto \frac{1}{-3} R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & g-6 \end{array}\right) \xrightarrow{R_3 \mapsto R_3 + 2R_2}$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & g-4 \end{array}\right) \xrightarrow{R_1 \mapsto R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & g-4 \end{array}\right) \Longrightarrow \text{corresponds to linear system} \begin{cases} x_1 = 1 \\ x_2 = 1 \\ 0 = g-4 \end{cases}$$

The latter is consistent iff $g=4$.

Answer: $g=4$ ∎

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## §1.4 The matrix equation $Ax = b$

We start from the following important definition which provides a convenient way to treat linear combinations of vectors

**Def:** If $A$ is an $m \times n$ matrix, with $n$ columns $\vec{a}_1, \ldots, \vec{a}_n$ (of height $m$), then the product of $A$ and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, denoted $A\vec{x}$, is the linear combination of the columns of $A$ with weights being entries of $\vec{x}$

$$A\vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_n \vec{a}_n$$

Example: $\begin{pmatrix} 1 & 3 & 10 \\ 2 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 29 \\ 20 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 10 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 3 \\ 10 \end{pmatrix} - 2 \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix}$$

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In particular, vector equation $x_1\vec{a_1} + x_2\vec{a_2} + \dots + x_k\vec{a_k} = \vec{b}$ is equivalent to a matrix equation $\boxed{A\vec{x} = \vec{b}}$

$$A = \begin{pmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_k} \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$$

**CONCLUSION:** If $A$ is an $m \times n$ matrix with columns $\vec{a_1}, \_\_, \vec{a_n}$, and $\vec{b} \in \mathbb{R}^m$, then the matrix equation $A\vec{x} = \vec{b}$ has the same solution set as the vector equation $x_1\vec{a_1} + \_\_ + x_n\vec{a_n} = \vec{b}$, which in turn has the same solution set as the linear system whose augmented matrix is $\left(\vec{a_1}\ \vec{a_2} \dots \vec{a_n} \mid \vec{b}\right)$

Note: The equation $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is a linear combination of the columns of $A$.

Ex 5: Given three vectors $\vec{V_1}, \vec{V_2}, \vec{V_3} \in \mathbb{R}^4$, assume $2\vec{V_1} + 3\vec{V_2} + 7\vec{V_3} = 0$.
Find $x_1, x_2$ such that $\begin{pmatrix} \vec{V_1} & \vec{V_2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{V_3}$

$$2\vec{V_1} + 3\vec{V_2} + 7\vec{V_3} = 0 \Rightarrow 7\vec{V_3} = -2\vec{V_1} - 3\vec{V_2} \Rightarrow \vec{V_3} = -\frac{2}{7}\vec{V_1} - \frac{3}{7}\vec{V_2}.$$

Thus: can take $x_1 = -2/7$, $x_2 = -3/7$

In practice, the product $A\vec{x}$ is computed using the Row-Vector Rule:

Row-Vector Rule

If the product $A\vec{x}$ is defined (i.e. size of $\vec{x}$ equals number of columns of $A$) then the $i^{th}$ entry of $A\vec{x}$ is the sum of products of the corresponding entries from the $i^{th}$ row of $A$ and from the vector $\vec{x}$

Examples:

$$\begin{pmatrix} 3 & 5 \\ -2 & 3 \\ 7 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\cdot 2 + 5\cdot 4 \\ (-2)\cdot 2 + 3\cdot 4 \\ 7\cdot 2 + 1\cdot 4 \end{pmatrix} = \begin{pmatrix} 26 \\ 8 \\ 18 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & 7 \\ 5 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + (-2)(-2) + 7\cdot 3 \\ 5\cdot 1 + 3\cdot(-2) + 1\cdot 3 \end{pmatrix} = \begin{pmatrix} 28 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

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**Properties:** 1) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

2) $A(c\vec{u}) = c\cdot(A\vec{u})$

Here,
- $\vec{u}, \vec{v} \in \mathbb{R}^n$
- $c \in \mathbb{R}$
- $A$ is an $m\times n$ matrix

(Explain if time remains
see Theorem 5 in §1.4)

Let us conclude today's lecture with the following very useful result

**CLAIM:** Let $A$ be an $m\times n$ matrix. Then the following are equivalent:
1) For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution
2) Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of $A$.
3) The columns of $A$ span $\mathbb{R}^m$ (i.e. their span equals $\mathbb{R}^m$)
4) $A$ has a pivot position in every row

Ex6: Given $\vec{v_1} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{v_2} = \begin{pmatrix}4\\5\\6\end{pmatrix}$, $\vec{v_3} = \begin{pmatrix}7\\8\\9\end{pmatrix}$ in $\mathbb{R}^3$, does $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ span $\mathbb{R}^3$?

▸ Consider the corresponding matrix

$$A = \begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix} \xrightarrow[R_3\mapsto R_3-3R_1]{R_2\mapsto R_2-2R_1} \begin{pmatrix}1&4&7\\0&-3&-6\\0&-6&-12\end{pmatrix} \xrightarrow{R_2\mapsto \frac{1}{-3}R_2} \begin{pmatrix}1&4&7\\0&1&2\\0&-6&-12\end{pmatrix} \xrightarrow[R_3\mapsto R_3+6R_2]{} \begin{pmatrix}1&4&7\\0&1&2\\0&0&0\end{pmatrix}$$

and there is no pivot position in the last row.

Thus, by above Claim, the answer is No! ▫

Ex7: Same question for $\vec{v_1} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{v_2} = \begin{pmatrix}4\\5\\6\end{pmatrix}$, $\vec{v_3} = \begin{pmatrix}7\\8\\10\end{pmatrix}$

▸ Answer is Yes! (carry out the calculations) ▫