Lecture #4

09/03/2020

- Last time → vector equation $x_1\vec{a_1} + x_2\vec{a_2} + \dots + x_k\vec{a_k} = \vec{b}$
  matrix equation $A\vec{x} = \vec{b}$
  } both equivalent to linear systems

  → Row-Vector rule for evaluation of $A\vec{x}$.

Ex1: Compute $\begin{pmatrix} 1 & 3 & 7 \\ 0 & 4 & 2 \\ -1 & -5 & -3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

► $\dots = \begin{pmatrix} 1\cdot 3 + 3\cdot(-2) + 7\cdot 1 \\ 0\cdot 3 + 4\cdot(-2) + 2\cdot 1 \\ (-1)\cdot 3 + (-5)(-2) + (-3)\cdot 1 \\ 2\cdot 3 + (-1)(-2) + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 4 \\ 9 \end{pmatrix}$ ■

As we saw last time, given explicitly an $m\times n$ matrix A and a vector $\vec{b}$ of height m, the problem of solving the matrix equation $A\vec{x} = \vec{b}$ boils down to solving linear system whose augmented matrix is $\left(A \mid \vec{b}\right)$ ← $m\times(n+1)$-matrix.

However, in some problems you will be explicitly given only an $m\times n$ matrix A, while $\vec{b}$ varies, and you will be asked if $A\vec{x}=\vec{b}$ admits a solution. To answer such questions we need the following general result:

CLAIM: Let A be an $m\times n$ matrix. Then the following are equivalent:
1) For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2) Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
3) The columns of A span $\mathbb{R}^m$ (i.e. their span equals $\mathbb{R}^m$)
4) A has a pivot position in each row

Ex2: Given $\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}$ in $\mathbb{R}^3$, does
$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ span $\mathbb{R}^3$
$\{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$ —11—

► No and Yes, respectively. See p.5 of Lecture #3 for details ■

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§1.5 Solution Sets of Linear Systems

As we discussed in Lecture 1, each linear system either doesn't have any solution, has exactly one solution, or infinitely many.

Question: What is the most convenient way to describe these solutions? (especially when there are infinitely many)

"Toy" Example: Solve $x+2y=1$

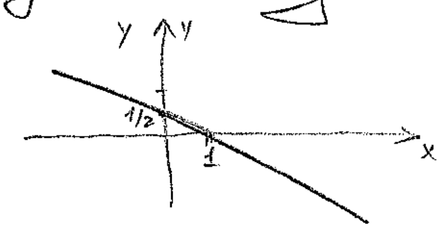
If we use our general machinery, we start by writing down the augmented matrix $(1\ 2\,|\,1)$, which is already reduced echelon. And as discussed in Lecture 1, we say $\begin{cases} y\text{-"free"} \\ x=1-2y \end{cases}$

But this can be equivalently written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

one can use any other parameter here, e.g. t

Note: geometrically $x+2y=1$ determines a line in the plane



while $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\cdot\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ provides a parametric equation of that line

But before we proceed to generalization, let's start from:

Def: A linear system is called homogeneous if it can be written in the form $A\vec{x}=\vec{0}$, where A is an $m\times n$ matrix and $\vec{0}\in\mathbb{R}^m$ – zero vector.

Note that any homogeneous linear system has the so-called trivial solution $\vec{x}=\vec{0}$.

Q: Are there any nontrivial solutions?

As follows from our discussions last week, we have:

CLAIM: The homogeneous equation $A\vec{x}=\vec{0}$ has a nontrivial solution iff this equation has at least one free variable

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Ex 3: Determine if the following homogeneous system has a nontrivial solution, and if it does - describe the solution set:

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 0 \\ 2x_1 - 5x_2 + 5x_3 = 0 \\ -3x_1 + 7x_2 - 8x_3 = 0 \end{cases}$$

► Augmented matrix $= \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -5 & 5 & 0 \\ -3 & 7 & -8 & 0 \end{array}\right) \xrightarrow[R_3 \mapsto R_3+3R_1]{R_2 \mapsto R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right)$

$\xrightarrow{R_3 \mapsto R_3+2R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \mapsto R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So: x_3 - free variable $\Rightarrow$ by above Claim there are nontrivial solutions

For any x_3, 2^{nd} equation implies $x_2 = -x_3$

1^{st} equation implies $x_1 = -5x_3$

Thus: $\vec{x} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}$

Note: Setting $x_3 = 0$, get the trivial solution

Q: What the solution set from Ex 3 looks like?

(Answer: it is a line through the origin in the direction $\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}$).

Ex 4: Same question for the system consisting of the single equation

$$x_1 - 3x_2 + 2x_3 = 0$$

► Augmented matrix $= (1 \;\; -3 \;\; 2 \,|\, 0)$ - already reduced echelon

Here: x_2, x_3 - "free" variables, while $x_1 = 3x_2 - 2x_3$

Thus: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

Q: What the solution set from Ex 4 looks like?

(Answer: Plane)

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Similarly to a line in $\mathbb{R}^2$, a plane in $\mathbb{R}^3$ can be described either via a single linear equation (sometimes called implicit) or via a parametric vector equation (sometimes called explicit) of the form

$$\boxed{\vec{x} = s\cdot\vec{u} + t\cdot\vec{v}}$$

where $\vec{u},\vec{v}$ - specified vectors in $\mathbb{R}^3$

s,t - parameters which vary over all real numbers.

E.g. in Ex4, we got $\vec{u} = \begin{pmatrix}3\\1\\0\end{pmatrix}$, $\vec{v} = \begin{pmatrix}-2\\0\\1\end{pmatrix}$, while x_2, x_3 - vary.

Note: It should be recalled that

1) a line in $\mathbb{R}^2$ can be also specified by two points on it
2) a plane in $\mathbb{R}^3$ can be also specified by three points on it, which do not lie on the same line.

So far we discussed only homogeneous linear systems, and we saw that their solution set can be always written as a span of several vectors!

Note: Even when the only solution is $\vec{0}$, still it equals Span$\{\vec{0}\}$

Our next question is: how to describe solutions of nonhomogeneous systems?

Before stating the general result, let us work out an example, compare to Ex 3:

Ex 5: Describe all solutions of

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 1 \\ 2x_1 - 5x_2 + 5x_3 = 3 \\ -3x_1 + 7x_2 - 8x_3 = -5 \end{cases}$$

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Augmentation matrix $= \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 2 & -5 & 5 & 3 \\ -3 & 7 & -8 & -5 \end{array}\right) \xrightarrow[R_3 \mapsto R_3+3R_1]{R_2 \mapsto R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \end{array}\right)$

$\xrightarrow{R_3 \mapsto R_3+2R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \mapsto R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ – reduced echelon

Here: x_3 – "free variable"

2^{nd} equation $\Rightarrow x_2 = 1 - x_3$

1^{st} equation $\Rightarrow x_1 = 4 - 5x_3$

Thus, writing the solution as a vector, we get

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4-5x_3 \\ 1-x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \underbrace{x_3 \begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}}$$

that was the solution of the corresponding homogeneous equation from Ex 3

Q: What the solution set from Ex 5 looks like?

(Answer: It's again a line parallel to $\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}$, but it no longer passes through the origin, instead it passes through the point $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$)

CLAIM: Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be any solution. Then, the solution set of the equation $A\vec{x} = \vec{b}$ has the form

$\boxed{\vec{p} + \text{any solution of the homogeneous eq-n } A\vec{x} = \vec{0}}$

Warning: Note that this claim does not apply to inconsistent equations.

Summarize the algorithm of writing down solution sets!