Lecture #5 09/08/2020

- Last time: 1) solutions sets of linear systems in parametric vector form
  2) solutions sets of nonhomogeneous linear systems via solutions sets of the corresponding homogeneous systems

Recall the results/strategy, but no examples on that to save time.

- §1.7 Linear Independence

Recall that linear systems with all constants featuring in their right-hand sides are called homogeneous. In the matrix form, they can be equivalently written as $A\vec{x}=0$

Example: $\begin{cases} x_1+4x_2+7x_3=0 \\ 2x_1+5x_2+8x_3=0 \\ 3x_1+6x_2+9x_3=0 \end{cases} \iff \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0.$

As we saw last time, $\vec{x}=\vec{0}$ is always a solution of homog. systems. The question is if there are any others.

Today: rephrase this a bit.

Recall $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

Hence: need to find weights $x_1, x_2, x_3$ so that the above linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ is a zero vector.

Hint: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \Rightarrow$ can take $x_1=1, x_2=-2, x_3=1$.

And actually all solutions have the form $\vec{x} = t\cdot\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(Ask why?)

But this discussion leads us to the following important definition

↓

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**Def**: An indexed set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\vec{v}_1 + \ldots + x_k\vec{v}_k = \vec{0}$$

has only the trivial solution.

Otherwise, the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be linearly dependent.

In the above example, we see that $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix} \in \mathbb{R}^3$ are linearly dependent.

From what we started, the following result is tautological:

**CLAIM**: The columns of a matrix $A$ are linearly independent iff the homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution.

Rmk: In practice, you will apply this result to determine if the given collection of vectors is linearly independent or not, by constructing matrix $A$ whose columns are the given vectors.

**Q**: Is it true that $\{\vec{v}\}$ is linearly independent for any $\vec{v} \in \mathbb{R}^n$.
set consisting of a single vector.

**A**: Yes, unless $\vec{v} = \vec{0}$ (as $x \cdot \vec{v} = \vec{0}$ with $x \neq 0$ iff $\vec{v} = \vec{0}$)

**Q**: Given any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, determine the criteria for $\{\vec{v}_1, \vec{v}_2\}$ to be linearly independent.

**A**: Neither of the two vectors is a multiple of another one.

The last question is closely related to the description of span of 2 vectors

**CLAIM**: 1) An indexed set $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ of two or more vectors is linearly dependent iff at least one of them is a linear combination of the others.

2) Furthermore, if $S$ is linearly dependent and $\vec{v}_1 \neq \vec{0}$, then for some $1 < j \leq k$, the vector $\vec{v}_j$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

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! <u>Warning</u>: the above Claim does <u>NOT</u> say that each vector $\vec{v}_j$ is a linear combination of the others – this is just <u>NOT TRUE</u>,
e.g. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ – linearly dependent, BUT $\begin{pmatrix}1\\1\end{pmatrix} \notin \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right\}$

! Note that if a set $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ contains the zero vector $\vec{0}$, then $S$ is linearly dependent (as $1 \cdot \vec{0} = \vec{0}$).

There is one case, when one can assert that $S$ is linearly dependent.

**CLAIM**: If $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, and $k > n$, then $S$ is linearly dependent

<u>Reason</u>: Boils down to the matrix equation $A\vec{x} = \vec{0}$, where $A$ is an $n \times k$ matrix. As $k > n$, there is at least one free variable $\Rightarrow$ there are nontrivial solutions.

<u>Ex 1</u>: Is the set of vectors $\left\{\begin{pmatrix}100\\-314\\2500\end{pmatrix}, \begin{pmatrix}\pi\\-e^2\\\sin(1)\end{pmatrix}, \begin{pmatrix}7000\\-1025\\\cos(\pi)\end{pmatrix}, \begin{pmatrix}1000000\\-\ln(105)\\e^{30}\end{pmatrix}\right\}$ linearly dependent or independent?

As there are 4 vectors in $\mathbb{R}^3$, $4>3$, this set is linearly dependent. But clearly finding a linear dependence relation is cumbersome.

<u>Ex 2</u>: Determine if the following sets are linearly dependent or not:

a) $\begin{pmatrix}1\\-2\\30\\4\end{pmatrix}, \begin{pmatrix}\pi\\\sin(1)\\e\\\ln(23)\end{pmatrix}, \begin{pmatrix}-3\\6\\-90\\-12\end{pmatrix}$

b) $\begin{pmatrix}1\\-2\\30\\4\end{pmatrix}, \begin{pmatrix}\pi\\\sin(1)\\e\\\ln(23)\end{pmatrix}, \begin{pmatrix}-3\\6\\-90\\-13\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

a) Lin. dependent as $3 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \vec{0}$ (equiv. $\vec{v}_3 = -3\vec{v}_1$)

b) Lin. dependent as this set contains $\vec{0}$

! <u>Rmk</u>: Let's conclude with observation that if columns of an $n \times k$ matrix span $\mathbb{R}^n$, then each row has a pivot $\Rightarrow$ # pivot columns $= n \Rightarrow n \le k$.

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Over the course of the previous 2 weeks you have seen that linear systems can be equivalently stated as vector equation $x_1\vec{v}_1 + \_\_ + x_k\vec{v}_k = \vec{w}$ or matrix eq-n $A\vec{x} = \vec{b}$

And all 3 questions were treated by the same machinery: reducing the augmented matrix to a reduced echelon form by an iterative application of elementary row operations.

SO: we have not seen any real benefit of using vector or matrix equations.

But the matrix equation $A\vec{x} = \vec{b}$, or better to say its left-hand side, brings us to a very important topic:

### §1.8 Linear Transformations

denoted $\overbrace{T: \mathbb{R}^n \longrightarrow \mathbb{R}^m}$

**Def**: A transformation (a.k.a. function or mapping) $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{x} \in \mathbb{R}^n$ a vector $T(\vec{x}) \in \mathbb{R}^m$

Here: $\mathbb{R}^n$ - domain of $T$

$\mathbb{R}^m$ - codomain of $T$

$T(\vec{x})$ - the image of $\vec{x}$ under the action of $T$

Set $\{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$ - the range of $T$.

For the rest of today, we are going to concentrate on those transformations $T$ which arise via matrix multiplication.

In other words, fix an $m \times n$ matrix $A$. It defines a matrix transformation

$$\boxed{\mathbb{R}^n \ni \vec{x} \longmapsto A\vec{x} \in \mathbb{R}^m}$$

Q: What is the domain, codomain, and range of this transformation?

A: Domain $= \mathbb{R}^n$

Codomain $= \mathbb{R}^m$

Range $= \{$all linear combinations of columns of $A\} = \text{span}\{$columns of $A\}$

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## Common simple questions (on matrix transformations $T(\vec{x}) = A\vec{x}$)

1) Compute image of $\vec{u} \in \mathbb{R}^n$: just evaluate $A \cdot \vec{u}$
2) Determine if $\vec{w}$ is in the range of $T$: check the matrix equation $A\vec{x} = \vec{w}$ for consistency
3) If $\vec{w} \in \mathbb{R}^m$ is in the range of $T$, find all $\vec{x} \in \mathbb{R}^n$ with image $= \vec{w}$: this boils down to a solution of the matrix equation $A\vec{x} = \vec{w}$ or equivalently the corresponding linear system.

Ex 3: Describe geometrically the following matrix transformations:

a) $\underset{\mathbb{R}^2}{\overset{\cap}{\vec{x}}} \longmapsto \underset{\mathbb{R}^2}{\overset{\cap}{A\vec{x}}}$ with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\underset{\mathbb{R}^3}{\overset{\cap}{\vec{x}}} \longmapsto \underset{\mathbb{R}^3}{\overset{\cap}{A\vec{x}}}$ with $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

c) $\underset{\mathbb{R}^2}{\overset{\cap}{\vec{x}}} \longmapsto \underset{\mathbb{R}^2}{\overset{\cap}{A\vec{x}}}$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

► a) Projects onto x-axis

b) Reflection in the xy-plane

c) $90°$-rotation counterclockwise around the origin $\blacksquare$

Recall the two properties of the matrix-vector product:

$$\boxed{\begin{aligned} A(\vec{u} + \vec{v}) &= A\vec{u} + A\vec{v} \\ A(c \cdot \vec{u}) &= c \cdot (A\vec{u}) \end{aligned}}$$

Thus, if $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$, then

$$\boxed{\begin{aligned} T(\vec{u} + \vec{v}) &= T(\vec{u}) + T(\vec{v}) \\ T(c\vec{u}) &= c \cdot T(\vec{u}) \end{aligned}} \quad \longleftarrow \text{for any } \vec{u}, \vec{v} \in \mathbb{R}^n \text{ and any } c \in \mathbb{R}$$

This brings us to the formal notion of linear transformations.

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**Def**: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

1) $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ for any $\vec{u},\vec{v} \in \mathbb{R}^n$
2) $T(c\vec{u}) = c\cdot T(\vec{u})$ for any $\vec{u} \in \mathbb{R}^n$, $c \in \mathbb{R}$

In other words, $T$ preserves operations of <u>vector addition</u> and <u>scalar multiplication</u>

Easy: $\boxed{\begin{array}{l} T(\vec{0}) = \vec{0} \\ T(c\vec{u}+d\vec{v}) = cT(\vec{u})+d\cdot T(\vec{v}) \text{ if } T\text{-linear} \end{array}}$ ← Discuss Reasons!

**Ex 4**: Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\5\\6\end{pmatrix}$$

Compute $T\begin{pmatrix}1\\2\end{pmatrix}$.

$$\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow T\begin{pmatrix}1\\2\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix}+2T\begin{pmatrix}0\\1\end{pmatrix} = \underbrace{\begin{pmatrix}1\\-3\\2\end{pmatrix}+2\begin{pmatrix}-2\\5\\6\end{pmatrix}}_{\begin{pmatrix}-3\\7\\14\end{pmatrix}}$$

$\blacksquare$

**Next time**: same reasoning will be used to show that any linear transformation $T$ is actually a matrix transformation.

**Note**:
$$\boxed{\begin{array}{c} T(\vec{v_1}+\vec{v_2}+\_\_+\vec{v_k}) = T(\vec{v_1})+T(\vec{v_2})+\_\_+T(\vec{v_k}) \\ \text{and even furthermore} \\ T(c_1\vec{v_1}+c_2\vec{v_2}+\_\_+c_k\vec{v_k}) = c_1T(\vec{v_1})+c_2T(\vec{v_2})+\_\_+c_kT(\vec{v_k}) \\ \text{if } T \text{ is linear} \end{array}}$$