Lecture #7 09/15/2020

- Last time $\longrightarrow$ Linear transformations = Matrix transformations
(recall how to recover the standard matrix)
$\longrightarrow$ Addition of Matrices & Scalar multiplication
(recall the basic properties involving these operations)

Q: If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator with the standard matrix A
$T_2 : \mathbb{R}^n \to \mathbb{R}^m$ — // — B,
construct linear operator $\mathbb{R}^n \to \mathbb{R}^m$ which corresponds to:
a) $A+B$
b) $c \cdot A$

A: a) $T = T_1 + T_2$, i.e. $T(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x})$
b) $T = c \cdot T_1$, i.e. $T(\vec{x}) = c \cdot T_1(\vec{x})$

Which other operations can be applied to linear operators?
We cannot multiply them clearly, but we can take compositions

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, dashed] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^k \end{tikzcd}$$

$\vec{x} \mapsto S(T(\vec{x}))$ $\leftarrow$ clearly it is a linear operator.
(check this!)

Hence, one can treat it as a matrix transformation.

Q: If the standard matrix of T is B (size $m \times n$) and the standard matrix of S is A (size $k \times m$), what is the standard matrix of their composition $(\vec{x} \mapsto S(T(\vec{x})))$?

A: If $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow T(\vec{x}) = B \cdot \vec{x} = \begin{pmatrix} \vec{b}_1 & \dots & \vec{b}_n \end{pmatrix} \cdot \vec{x} = x_1 \vec{b}_1 + \dots + x_n \vec{b}_n$

$\Rightarrow S(\vec{x}) = S(x_1 \vec{b}_1 + \dots + x_n \vec{b}_n) = x_1 S(\vec{b}_1) + \dots + x_n S(\vec{b}_n) = x_1 \cdot (A \cdot \vec{b}_1) + \dots + x_n (A \cdot \vec{b}_n)$

and the latter can be written as

$$\underbrace{\begin{pmatrix} A\vec{b}_1 & \dots & A\vec{b}_n \end{pmatrix}} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

the looked-after matrix

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This brings us to the following:

Def: If A is an $k\times m$ matrix, B is an $m\times n$ matrix, then the product AB is the $k\times n$ matrix whose columns are $A\vec{b_1}, \dots, A\vec{b_n}$:

$$AB = \left(A\vec{b_1} \ \dots \ A\vec{b_n} \right)$$

Ex 1: Compute

a) $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot1+2\cdot2+3\cdot4 & 1\cdot(-2)+2\cdot1+3\cdot0 \\ -4\cdot1+0\cdot2+6\cdot4 & -4\cdot(-2)+0\cdot1+6\cdot0 \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 20 & 8 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & 6 \end{pmatrix}$

Note: For the product $A\cdot B$ to be well-defined, the number of columns in A must coincide with the number of rows in B.

Evoking the matrix-column rule for computation of $A\vec{b}$, we get

$$\boxed{(A\cdot B)_{ij} = a_{i1}\cdot b_{1j} + a_{i2}\cdot b_{2j} + \dots + a_{in}\cdot b_{nj}}$$

↑ row-column rule for computing $A\cdot B$

(reads: to find the $(i,j)^{th}$ entry of $A\cdot B$, take the sum of products of elements in the i^{th} row of A with corresponding elements in the j^{th} column of B.)

Ex2: Compute $\begin{pmatrix} 100 & 0 & 0 \\ 0 & -10 & 0 \\ e^2 & 204 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 19 & 17 & 134 \end{pmatrix} = 0$

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Properties of Matrix Multiplication

Let A be an $m\times n$ matrix and B, C be two matrices of appropriate size.

1) $A(BC) = (AB)C$
2) $A(B+C) = AB + AC$
3) $(B+C)A = BA + CA$
4) $r\cdot(AB) = (rA)\cdot B = A\cdot(rB) \quad r\in\mathbb{R}$
5) $I_m \cdot A = A = A\cdot I_n$, where $I_k = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{k} \Big\updownarrow k$ - the identity matrix

__Upshot__: The addition and product of matrices satisfy the same rules as real number, in particular, can open parentheses in a usual way.

! __However__: $AB \neq BA$ even when both products are well-defined.

__Ex3__: Compute AB and BA for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 7 \\ 1 & 6 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & 19 \\ -4 & 51 \end{pmatrix} \quad BA = \begin{pmatrix} 18 & 29 \\ 19 & 32 \end{pmatrix}$$. Clearly $AB \neq BA$

! __More Warnings__:

1) If $AB = 0$, it is __not__ true that $A=0$ or $B=0$, see Ex2
2) If $AB = AC$, it is __not__ true that $B = C$

__Q__: For which matrices $A\cdot A$ makes sense?

__Def__: Given an $n\times n$ matrix A and positive integer $k>0$, define __k-th power of A__ as $A^k := \underbrace{A\cdot A\cdot \ldots \cdot A}_{k \text{ times}}$

__Convention__: $A^0 := I_n$

__Q__: $A^k \cdot A^l = ?$ $\leftarrow A^{k+l}$

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There is one more operation on matrices that will come handy later on:

Def: Given an $m\times n$ matrix A, the transpose of A, denoted A^T, is the $n\times m$ matrix whose columns are formed from corresponding rows of A

Examples: $\begin{pmatrix}1\\3\\-5\end{pmatrix}^T = (1\ 3\ -5)$, $\begin{pmatrix}1 & -4\\-3 & 2\\6 & -7\end{pmatrix}^T = \begin{pmatrix}1 & -3 & 6\\-4 & 2 & -7\end{pmatrix}$

the transpose of a column-vector is a row-vector with the same entries.

Q: Is there a nonzero column vector $\vec{x}\in\mathbb{R}^n$ such that $\vec{x}^T\cdot\vec{x}=0$?

A: If $\vec{x}=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$, then $\vec{x}^T\cdot\vec{x}=x_1^2+\_\_\_+x_n^2>0$ for $\vec{x}\neq\vec{0}$. Hence, NO ■

Q: If A is an $k\times m$ matrix, and B is an $m\times 3$ matrix whose last column is the sum of the first two. What can be said about AB?

A: Its 3^{rd} column is also the sum of the first two ■

Finally, there is one more operation, the inverse, defined for some matrices. And this brings us to:

§2.2 The inverse of a matrix

Def: An $n\times n$ matrix A is called invertible if there is another $n\times n$ matrix C s.t

$$C\cdot A = I_n = A\cdot C$$

the $n\times n$ identity matrix

Q: Can there be several matrices C satisfying these properties?

No: If $C_1\cdot A = I_n = A\cdot C_2$, then $C_1 = C_1(AC_2)=(C_1A)C_2=C_2 \Rightarrow C_1=C_2$ ■

So: If C exists, it is unique and is denoted by A^{-1}.

inverse matrix

Ex 4: Does the matrix $\begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}$ admit an inverse?

No: for any $C=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ get $\begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}\cdot\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}* & *\\ * & 0\end{pmatrix}$ not 1! ■

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Def: An $n \times n$ matrix A is called *singular* if it is not invertible.

The inverse of 2×2 matrices is particularly simple:

CLAIM: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix.

1) A is not invertible iff $ad - bc = 0$
2) If $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Def: The above quantity $\det A := ad - bc$ is called the *determinant* of A.

Ex 5: Compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$

▶ $A^{-1} = \frac{1}{1}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$ ■

Important Observation: If A is an invertible $n \times n$ matrix, then for any $\vec{b} \in \mathbb{R}^n$ there is a unique solution of $A\vec{x} = \vec{b}$, given explicitly by $\vec{x} = A^{-1}\vec{b}$

Properties of Inverse

1) If A is invertible, then A^{-1} is also invertible with
$$(A^{-1})^{-1} = A$$
2) If A, B are invertible $n \times n$ matrices, then so is AB with
$$(AB)^{-1} = \underbrace{B^{-1} \cdot A^{-1}}$$
Note the opposite order in the product
3) If A is invertible, then so is A^T with
$$(A^T)^{-1} = (A^{-1})^T$$

Properties of Transpose ← should be mentioned on p. 4!

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(rA)^T = r \cdot A^T$
4) $(AB)^T = \underbrace{B^T \cdot A^T}$
the opposite order in the product