# Lecture #8

- Last time → Multiplication of matrices
  - ! Corresponds to composition of linear transformations
  - → Transpose (reverts rows ⟷ columns)
  - → Inverse (not always defined)

## Some points to keep in mind

1) $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1} A^{-1}$ ← reversed order in both

2) $AB \neq BA$

Q: Given a square $n \times n$ matrix $A$ and a value of $\overset{n \times m \text{ matrix}}{A \cdot B}$, recover $\overset{n \times m \text{ matrix}}{B}$
  - Invert $A$ to find $A^{-1}$, then $B = A^{-1} \cdot (AB)$. 

Q: Same question but you are given an $m \times n$ matrix $BA$ instead of $AB$
  - Then $B = (BA) \cdot A^{-1}$. 

Q: For any $n \times n$ matrix $A$ can you find a non-zero & non-identity matrix $B$ such that $AB = BA$?
  - Can take $B = A, A^2, \underline{\quad}$. 

Ex1: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, so that $A$ is non-invertible as $\det(A) = 1 \cdot 6 - 2 \cdot 3 = 0$.
  Find a $2 \times 2$ matrix $B$ so that $AB = 0$
  - Let $B = \begin{pmatrix} \vec{b_1} & \vec{b_2} \end{pmatrix}$. Then both $\vec{b_1}, \vec{b_2}$ are solutions of $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Clearly all solutions are multiples of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
  Hence, can take e.g. $B = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$

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- ## Properties of inverse

  1) A - invertible $\Rightarrow$ $A^{-1}$ - invertible : $\boxed{(A^{-1})^{-1} = A}$

  2) A, B - invertible $n \times n$ matrices $\Rightarrow$ AB - invertible: $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$

  3) A - invertible $\Rightarrow$ $A^T$ - invertible . $\boxed{(A^T)^{-1} = (A^{-1})^T}$

CLAIM : If A is an invertible $n \times n$ matrix, then for any $\vec{b} \in \mathbb{R}^n$ there is a unique solution of $A\vec{x} = \vec{b}$, given explicitly by $\vec{x} = A^{-1} \vec{b}$

Question: How to compute $A^{-1}$ (know the $2 \times 2$ case so far)?

CLAIM: An $n \times n$ matrix A is invertible iff A is row-equivalent to $I_n$. In this case, any sequence of elementary row operations that reduces A to $I_n$, also transforms $I_n$ to $A^{-1}$.

In order not to do the same work twice, the above claim can be interpreted as the following algorithm to find inverse $A^{-1}$:

Algorithm: Row reduce the $n \times 2n$ matrix $(A \mid I_n)$.
    If A is row equivalent to $I_n$, then $(A \mid I_n)$ is row equivalent to $(I_n \mid A^{-1})$. Otherwise, A is singular

Another perspective to this algorithm

The row reduction of $(A \mid I_n)$ to $(I_n \mid A^{-1})$ can be viewed as a simultaneous solution of $n$ vector equations
$$A\vec{x} = \vec{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A\vec{x} = \vec{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \longrightarrow \quad A\vec{x} = \vec{e_n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

But by definition of the inverse and product, we exactly know that the $i^{th}$ column of B is a solution of the $i^{th}$ equation above.

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## §2.3 Characterizations of invertible matrices

The notion of "invertible matrix" is closely related with previously discussed notions & questions due to the following important result:

> **CLAIM:** Let $A$ be an $n \times n$ matrix. The following are equivalent:
>
> 1) $A$ - invertible
>
> 2) $A$ - row equivalent to $I_n$
>
> 3) $A$ has $n$ pivot positions
>
> 4) $A\vec{x} = \vec{0}$ has only the trivial solution
>
> 5) The columns of $A$ form a linearly independent set
>
> 6) The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one
>
> 7) The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
>
> 8) The columns of $A$ span $\mathbb{R}^n$.
>
> 9) The equation $A\vec{x} = \vec{b}$ has at least $\left(\begin{smallmatrix}\text{and actually}\\\text{exactly}\end{smallmatrix}\right)$ one solution for any $\vec{b} \in \mathbb{R}^n$.
>
> 10) There is an $n \times n$ matrix $C$ such that $CA = I_n$
>
> 11) There is an $n \times n$ matrix $D$ such that $AD = I_n$
>
> 12) $A^T$ is invertible

Comment on 10) & 11): If $CA = I_n \overset{\text{CLAIM}}{\Rightarrow} A$-invertible $\Rightarrow C = A^{-1}$
$AD = I_n \overset{\text{CLAIM}}{\Rightarrow} D = A^{-1}$ $\Bigg\} \Rightarrow$

> **CLAIM:** If $A, B$ are $n \times n$ matrices satisfying $AB = I_n$, then $A, B$ - invertible with $A^{-1} = B$ and $B^{-1} = A$

❗ Most efficient way (so far) is to apply 2) or 3) to decide if $A$ is invertible or not.

Ex 2: Is the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 17 \end{pmatrix}$ invertible?

$A \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -35 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ \cdots \end{pmatrix} \Rightarrow$ not invertible as we got zero row!

❗ The above Claim refers only to square matrices.

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Q: What is the meaning of this notion "invertible" in terms of the corresponding linear transformations $\mathbb{R}^n \to \mathbb{R}^n$?

First, let me give a definition:

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(S(\vec{x})) = \vec{x} = S(T(\vec{x})) \text{ for any } \vec{x} \in \mathbb{R}^n. \quad (*)$$

CLAIM: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, $A$ — its standard matrix. Then, $T$ — invertible iff $A$ — invertible matrix. In that case, the linear transformation $S: \vec{x} \mapsto A^{-1}\vec{x}$ is the unique solution $S$ satisfying $(*)$

So: $S$ is unique & linear (if it exists).

Ex 3: a) Is it true that a one-to-one linear transformation $\mathbb{R}^1 \to \mathbb{R}^3$ is onto?

b) Is it true that a one-to-one linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ is onto?

a) No, e.g. $T: x \mapsto \begin{pmatrix} x \\ x \\ x \end{pmatrix}$ is one-to-one, but $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ is not in the range.

b) Yes: by Claim on p.3: the standard matrix is invertible $\Rightarrow T$ is onto

Ex 4: Assume $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(\vec{e_1}) = T(3\vec{e_2})$. Can $T$ be onto?

$T(\vec{e_1}) = T(3\vec{e_2}) \Rightarrow T$ is not one-to-one $\Rightarrow$ its standard matrix $A$ is not invertible

$T$ is not onto

Let us now move to a new concept which is closely related with matrices and matrix equations.

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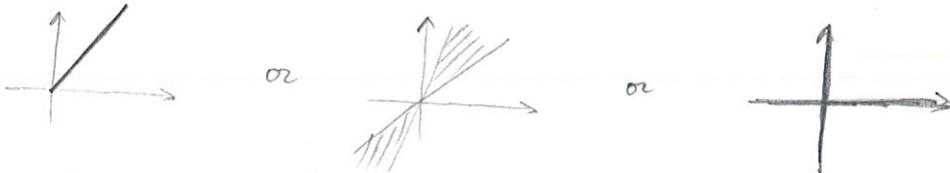## §2.8 Subspaces of $\mathbb{R}^n$

**Def:** A set $H$ of $\mathbb{R}^n$ is called a _subspace of $\mathbb{R}^n$_ if the following holds:

> 1) $\vec{0} \in H$
>
> 2) If $\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H$
>
> 3) If $\vec{u} \in H \Rightarrow c \cdot \vec{u} \in H$ for any $c \in \mathbb{R}$

Examples: $\{\vec{0}\}$, line $\{\mathbb{R} \cdot \vec{v}\}$, coordinate plane $\mathbb{R}^2_{x_1 x_2} \subseteq \mathbb{R}^3_{x_1 x_2 x_3}$

Counter examples:



Generalizing the above examples, we see that

> Given any collection $\{\vec{v_1}, \ldots, \vec{v_k}\}$ of vectors in $\mathbb{R}^n$, their span Span $\{\vec{v_1}, \ldots, \vec{v_k}\}$ is a subspace of $\mathbb{R}^n$

! The converse is also true — shall be discussed below.

Let's now discuss two constructions producing subspaces given a matrix

**Def:** The _column space_ of a matrix $A$, denoted $\text{Col } A$, is the set of all linear combinations of the columns of $A$ (i.e. the span of columns)

For an $m \times n$ matrix $A \rightsquigarrow \text{Col } A$ is thus a subspace of $\mathbb{R}^m$

(! $\text{Col } A$ coincides with the range of the corresponding linear transformation $\mathbb{R}^n \to \mathbb{R}^m$)

**Def:** The _null space_ of a matrix $A$, denoted $\text{Nul } A$, is the set of all solutions of the homog. equation $A\vec{x} = \vec{0}$

Clearly, given an $m \times n$ matrix $A$, $\text{Nul } A$ is a subspace of $\mathbb{R}^n$.

Note: while $\text{Col } A$ is defined explicitly, $\text{Nul } A$ is defined above quite implicitly. To get an explicit description of $\text{Nul } A$, one needs to write sol-ns of $A\vec{x} = \vec{0}$ in parametric form.

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**Def:** A basis of a subspace $H$ of $\mathbb{R}^n$ is a linearly independent set in $H$ which spans $H$.

Regarding the above two constructions, we have:

> **CLAIM 1:** Solving homogeneous matrix equation $A\vec{x} = \vec{0}$ in the parametric vector form automatically yields a basis in Nul $A$.

> **CLAIM 2:** The pivot columns of a matrix $A$ form a basis of Col $A$.

! Note that we say "a basis" as there are too many ways to pick one.

**Ex 5:** For $A = \begin{pmatrix} 1 & 2 & 7 & 10 & 12 \\ 2 & 5 & 8 & 11 & 6 \\ 4 & 6 & 9 & 5 & 0 \end{pmatrix}$ find a basis of Col $A$ and Nul $A$.