Lecture #9

09/22/2020

- **Last time** → Explicit algorithm for computing A^{-1}
  - → Characterization of invertible matrices (12 criteria & will have some more soon)
  - → Useful corollary that "left inverse" = "right inverse" = "inverse" i.e. if $AB = I_n$ or $BA = I_n$, then actually both hold and $B = A^{-1}$, where A was an $n\times n$ matrix
  - → invertibility of an $n \times n$ matrix $=$ invertibility of the corresponding linear transformation $\mathbb{R}^n \to \mathbb{R}^n$.
  - → Subspaces → Null space, Column space, Basis

Ex 1: Evaluate the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \mapsto R_3 - 5R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \mapsto R_3 + 4R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 5 & -5 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \mapsto \frac{1}{5} R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{4}{5} & \frac{1}{5} \end{array}\right) \xrightarrow[R_1 \mapsto R_1 - 3R_3]{R_2 \mapsto R_2 - 5R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -\frac{12}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -1 & 4/5 & 1/5 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & \frac{18}{5} & \frac{7}{5} \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -1 & 4/5 & 1/5 \end{array}\right)$$

So: $A^{-1} = \begin{pmatrix} -6 & \frac{18}{5} & \frac{7}{5} \\ 5 & -3 & -1 \\ -1 & 4/5 & 1/5 \end{pmatrix}$

Q: How would you solve equation $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{pmatrix} \cdot B = \begin{pmatrix} 0 & -5 \\ 10 & 15 \\ 20 & 0 \end{pmatrix}$? (multiply on the left by A^{-1} from Ex1)

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Ex 2: Are the following matrices invertible or not?

a) $A = \begin{pmatrix} 1 & -2 & 7 \\ 2 & -4 & 11 \\ 7 & -14 & 18 \end{pmatrix}$ No: 2^{nd} column $= -2 \cdot 1^{st}$ column

b) $A = \begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & 0 & 0 & 0 \\ 3 & -6 & 13 & -111 \\ -2 & 7 & 11 & 15 \end{pmatrix}$ No: As the columns do not span the entire $\mathbb{R}^4$, since any linear combination of these 4 columns has second coordinate $=0$.

c) $A = \begin{pmatrix} 1 & 2 & 5 & 12 \\ 1 & 3 & 0 & 13 \\ 2 & 0 & 0 & 14 \\ 0 & 0 & 0 & 16 \end{pmatrix}$ Yes: It's quite obvious that columns span $\mathbb{R}^4$.

Ex 3: Given 3 vectors $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \in \mathbb{R}^3$,

determine if they form a basis of $\mathbb{R}^3$ or not.

▶ Recalling the definition of "basis", we need to verify:

1) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ – linearly independent set

2) any vector in $\mathbb{R}^3$ is their linear combination.

But from last class we know that both are equivalent to invertibility of $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 4 \\ 1 & 6 & -5 \end{pmatrix}$. To check the latter, reduce to echelon form:

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 4 \\ 1 & 6 & -5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & -5 \\ 3 & 0 & 4 \\ 2 & 1 & 3 \end{pmatrix} \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - 3R_1} \begin{pmatrix} 1 & 6 & -5 \\ 0 & -18 & 19 \\ 0 & -11 & 13 \end{pmatrix} \xrightarrow{R_2 \mapsto -\frac{1}{18} R_2} \begin{pmatrix} 1 & 6 & -5 \\ 0 & 1 & -19/18 \\ 0 & -11 & 13 \end{pmatrix}$$

$$\xrightarrow{R_3 \mapsto R_3 + 11R_2} \begin{pmatrix} 1 & 6 & -5 \\ 0 & 1 & -19/18 \\ 0 & 0 & \underbrace{13 - \frac{209}{18}}_{\text{pivot positions}} \neq 0 \end{pmatrix} \Rightarrow A\text{-invertible} \Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ form a basis.}$$

Q: Why there are no one-to-one linear maps $\mathbb{R}^4 \to \mathbb{R}^3$?
(Use that any set of $> n$ vectors in $\mathbb{R}^n$ is linearly dependent)

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Ex 4: For $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix}$ find a basis of Col A and Nul A.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & -3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \underline{1} & 1 & 2 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, the pivot columns are #1, 2 $\Rightarrow$ the 1^{st} & 2^{nd} columns of A form a basis of Col A.
$\uparrow \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $\uparrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

[Indeed, they are clearly lin. indep as one is not a multiple of another, while the 3^{d} column is the sum of 1^{st} & 2^{nd}]

On the other hand, solving homog. eq-n $A\vec{x} = 0$, we see that x_3 - free variable $\Rightarrow x_2 = -x_3,\ x_1 = -x_3 \Rightarrow$ all solutions have form
$\begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Hence, $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis of Nul A.

[One could alternatively take any nonzero multiple of $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$].

Warning: Any non-zero space has infinitely many bases.
In particular, we could also say that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -4 \\ -5 \end{pmatrix} \right\} \text{ - bases of Col A}$$

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§2.9 Dimension and Rank

The notion of a basis allows one to introduce coordinates:

Def: Suppose $\mathcal{B}=\{\vec{b}_1, \ldots, \vec{b}_k\}$ is a basis of a subspace $H \subseteq \mathbb{R}^n$.
For each $\vec{x} \in H$, the coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, \ldots, c_k$ such that

$$\boxed{\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \ldots + c_k \cdot \vec{b}_k}$$

The vector

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

is called the coordinate vector of $\vec{x}$ (relative to $\mathcal{B}$) or the $\mathcal{B}$-coordinate vector of $\vec{x}$.

Example: If $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 1 \\ 8 \\ 11 \end{pmatrix}$, then $\vec{x} = 3\vec{v}_1 - \vec{v}_2 \Rightarrow \vec{x} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$.
Clearly $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ - a basis of $\text{Span}\{\vec{v}_1, \vec{v}_2\}$, and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

! Observation: The map $\begin{array}{l} H \to \mathbb{R}^k \\ \vec{x} \mapsto [x]_{\mathcal{B}} \end{array}$ is a one-to-one correspondence, i.e. the elements of the subspace $H \subseteq \mathbb{R}^n$ are parametrized by $\mathbb{R}^k$.

CLAIM: Any basis of a subspace H has the same number of elements

Def: The dimension of a non-zero subspace H, denoted $\dim(H)$, is the number of vectors in any basis of H.

! By default, the dimension of the zero subspace $\{\vec{0}\}$ is defined to be zero.

Example: In Ex4, $\dim(\text{Col } A) = 2$ and $\dim(\text{Null } A) = 1$.

Def: The rank of a matrix A, denoted $\text{rank } A$, is the dimension of $\text{Col } A$.

As the pivot columns form a basis of $\text{Col } A$, we obtain

$$\boxed{\text{rank } A = \#\text{ pivot columns of } A}$$

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Q: What about the dimension of Nul A?

A: As follows from last class, we have

$$\boxed{\dim \text{Nul } A = \# \text{ nonpivot columns}}$$

As each column of A is either a pivot or nonpivot column, we get

CLAIM: rank A + dim Nul A = # columns of A.

Evoking the aforementioned one-to-one correspondence b/w H & $\mathbb{R}^k$ as well as the criteria for invertibility of a $k \times k$ matrix, one arrives at:

CLAIM: Let $H \subseteq \mathbb{R}^n$ be a k-dimensional subspace. Any linearly independent set of exactly k elements of H is a basis of H. Likewise, any set of k elements of H that spans H is a basis of H.

Using the above concepts, we can now extend the list of criteria for an invertibility of an $n \times n$ matrix:

CLAIM: Let A be an $n \times n$ matrix. The following are equivalent:

1) A - invertible
2) The columns of A form a basis of $\mathbb{R}^n$
3) Col $A = \mathbb{R}^n$
4) rank $A = n$
5) dim Nul $A = 0$
6) Nul $A = \{\vec{0}\}$

Q: Does there exist a 3×4 matrix A with rank $A = 4$.

(that's just a rephrased Q from the end of p.2).