Lecture #10

§3.1 Determinants

Recall that when computing the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we introduced

$$\det A = ad - bc.$$

In particular, a 2×2 matrix A is invertible iff $\det A \neq 0$.

Following our general notations, we can rephrase the above definition as:

Def: The determinant of a 2×2 matrix $(a_{ij})_{i,j=1}^{2}$ is

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := a_{11}a_{22} - a_{12}a_{21}$$

For $n>2$, the determinant of an $n\times n$ matrix $(a_{ij})_{i,j=1}^{n}$ is defined recursively

Def: The determinant of $A = (a_{ij})_{i,j=1}^{n}$ is defined as an alternating sum

$$\boxed{\det A = a_{11}\det A_{11} - a_{12}\cdot \det A_{12} + a_{13}\cdot \det A_{13} - \ldots + (-1)^{n-1} a_{1n} \det A_{1n}}$$

where A_{ij} is an $(n-1)\times(n-1)$ matrix obtained from A by deleting i^{th} row & j^{th} column.

Convention-wise: for a 1×1 matrix $A = (a_{11})$, have $\det A := a_{11}$.

"Expansion across the first row formula".

Ex 1: Compute the determinant of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{pmatrix}$ ← from Ex 1 of Lecture 9

$$|A| = 1\cdot(1\cdot 0 - 5\cdot 6) - 2(0\cdot 0 - 5\cdot 5) + 3\cdot(0\cdot 6 - 1\cdot 5) = -30 + 50 - 15 = 5$$

Def: Given an $n\times n$ matrix A, the $(i,j)^{th}$ cofactor of A is defined via

$$\boxed{C_{ij} := (-1)^{i+j}\cdot \det A_{ij}}$$

Then, the above definition of $\det A$ can be written as

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \_\_ + a_{1n}C_{1n}}$$

But one can use expansion along any row or column to compute $\det(A)$!

Lecture # 10

CLAIM: Given an $n\times n$ matrix $A=(a_{ij})_{i,j=1}^{n}$, its determinant can be computed by a cofactor expansion across any row or down any column:

1) The expansion across the i^{th} row gives:

$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}}$$

2) The expansion down the j^{th} column gives:

$$\boxed{\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}}$$

This choice is particularly important when A has a lot of 0's.

Ex2: Compute the determinant of $B=\begin{pmatrix} 2 & 7 & 4 & 8 & 13 & 17 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 7 & 4 & 7 & 11 & 13 \\ 0 & 8 & 0 & 1 & 2 & 3 \\ 0 & 9 & 0 & 0 & 1 & 5 \\ 0 & 12 & 0 & 5 & 6 & 0 \end{pmatrix}$

$$\det B \xlongequal[\text{expansion}]{1^{st}\text{ column}} 2\cdot\begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ 7 & 4 & 7 & 11 & 13 \\ 8 & 0 & 1 & 2 & 3 \\ 9 & 0 & 0 & 1 & 5 \\ 12 & 0 & 5 & 6 & 0 \end{vmatrix} \xlongequal[\text{expansion}]{1^{st}\text{ row}} 2\cdot 3\cdot\begin{vmatrix} 4 & 7 & 11 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 5 & 6 & 0 \end{vmatrix}$$

$$\xlongequal[\text{expansion}]{1^{st}\text{ column}} 2\cdot 3\cdot 4\cdot\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{vmatrix} \xlongequal{Ex1} 2\cdot 3\cdot 4\cdot 5 = \boxed{120}$$

The same method immediately leads to the following result:

Claim: If an $n\times n$ matrix $A=(a_{ij})$ is triangular, i.e. $A=\begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & a_{33} & \\ 0 & & & \ddots \ a_{nn} \end{pmatrix}$

then $\det A = a_{11}\cdot a_{22}\cdot\dots\cdot a_{nn}$

Lecture #10

However, in the majority of cases, the matrix A will not have any zero entries at all, hence, any choice of row/column cofactor expansion will take quite a lot of time.

Question: Is there any faster way to evaluate det A?

It turns out that the key idea is again to perform the elementary row transformations to A, reducing it to an echelon form. The latter is based on the following result:

CLAIM: Let A be a square matrix.

a) If a multiple of one row is added to another row to produce a matrix B, then $\det A = \det B$

b) If two rows of A are interchanged to produce B, then $\det B = -\det A$

c) If one row is multiplied by k to produce B, then $\det B = k \cdot \det A$

Ex3: Compute det A for $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 8 \\ -3 & -4 & 11 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & -2 & 3 \\ 2 & -4 & 8 \\ -3 & 4 & 11 \end{vmatrix} \xlongequal[R_3 \mapsto R_3 + 3R_1]{R_2 \mapsto R_2 - 2R_1} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 0 & 2 \\ 0 & -2 & 20 \end{vmatrix} \xlongequal{R_2 \leftrightarrow R_3} -\begin{vmatrix} 1 & -2 & 3 \\ 0 & -2 & 20 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\xlongequal{\text{det of triangular matrix}} -1 \cdot (-2) \cdot 2 = \boxed{4}$$

Ex4: Use part c) to express $\det(k \cdot A)$ via $\det A$ for an $n \times n$ matrix A.

Applying c) n times, get $\det(k \cdot A) = k^n \cdot \det A$

Lecture #10

As we know, performing only elementary row operations from a) & b), one can reduce matrix A to an echelon matrix U. Thus:

$$\boxed{\text{Claim: } \det A = (-1)^r \cdot \det U, \text{ where } r = \#\text{interchanges}}$$

Being in echelon form, U must be triangular. Hence if $U = (u_{ij})_{i,j=1}^{n}$, then $\boxed{\det U = u_{11} \cdot u_{22} \cdot \ldots \cdot u_{nn} \Rightarrow \det(A) = (-1)^r \cdot u_{11} \cdot u_{22} \cdot \ldots \cdot u_{nn}}$

Moreover, being an $n \times n$ matrix, we know that

- if A is invertible $\Rightarrow$ all $u_{11}, u_{22}, \ldots, u_{nn}$ – pivots
- if A is singular $\Rightarrow u_{nn} = 0$

Therefore, we conclude:

$$\boxed{\det A = \begin{cases} 0, & \text{if } A \text{ is singular} \\ (-1)^r \cdot \underset{\text{of pivots}}{\text{product}}, & \text{if } A \text{ is invertible} \end{cases}}$$

In particular, we get one more criteria of invertibility of a matrix

$$\boxed{\text{CLAIM: A square matrix } A \text{ is invertible iff } \det(A) \neq 0}$$

Another interesting corollary is:

$$\boxed{\begin{array}{l}\text{Claim: While the echelon form } U \text{ of } A \text{ is not unique,} \\ \text{the product of the pivots is unique, up to a sign.}\end{array}}$$

Ex 5: Compute $\det A$ for $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & \underset{-(c-a)(b+a)}{c^2-a^2} \end{vmatrix}$$

$$= (b-a)(c^2 - \cancel{a^2} - bc + ab - ac + \cancel{a^2}) = (b-a)(c-b)(c-a)$$

Q: Any guess about the formula for $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$ etc. ?

Lecture #10

Evoking "A - invertible" $\Leftrightarrow$ "columns of A are linearly independent" we see that

$$\boxed{\det A = 0 \text{ iff columns of } A \text{ are linearly dependent}}$$

Ex6: For which values of a, b, c the vectors $\begin{pmatrix}1\\a\\a^2\end{pmatrix}, \begin{pmatrix}1\\b\\b^2\end{pmatrix}, \begin{pmatrix}1\\c\\c^2\end{pmatrix}$ form a basis of $\mathbb{R}^3$?

▶ As follows from Ex5, they form a basis iff $\begin{cases} a \neq b \\ a \neq c \\ b \neq c \end{cases}$ i.e. a, b, c are pairwise distinct ∎

! In practice, you may often wish to combine row operations together with row/column cofactor formulas.

CLAIM (see p.182): For any $n \times n$ matrix A, we have

$$\det A = \det \underbrace{A^T}_{\text{transposed of } A}$$

But the "elementary row operations" on the transposed side yield the "elementary column operations". Thus:

Claim: Let A be a square matrix.

a) If a multiple of one column is added to another column to produce a matrix B, then
$$\det B = \det A$$

b) If two columns of A are interchanged to produce B, then
$$\det B = -\det A$$

c) If one column is multiplied by k to produce B, then
$$\det B = k \cdot \det A$$

Lecture #10

Ex 7: Given $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = 7$, compute $\det B$ for

1) $B = \begin{pmatrix} a & 7c & 5b+a \\ d & 7f & 5e+d \\ g & 7k & 5h+g \end{pmatrix}$

2) $B = 3\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$

1) $\det B = 7 \cdot 7 \cdot 5 \cdot (-1)$

2) $\det B = 3 \cdot 3 \cdot 3 \cdot 7$ ☐

One more important property of determinants is:

Claim: Given two $n \times n$ matrices A & B, we have
$$\det(A \cdot B) = \det A \cdot \det B$$

Warning: $\det(A+B) \neq \det A + \det B$

Ex 8: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{pmatrix}$ and $B = A^4$. Find $\det B$.

$$\det B = \det(A \cdot A \cdot A \cdot A) = (\det A)^4 \underset{\text{Ex 1}}{=} 5^4 = 625$$ ☐

Ex 9: Let A and B be 3×3 matrices with $\det A = 2$, $\det B = -3$.

a) Compute $\det(A^T B A B^T)$

b) Compute $\det(B^{-2} \cdot A^3 \cdot B^4)$

a) $2 \cdot (-3) \cdot 2 \cdot (-3) = 36$

b) $(-3)^{-2} \cdot 2^3 \cdot (-3)^4 = 72$ ☐