Lecture #13 10/06/2020

• § 4.1 Vector Spaces and Subspaces

Today we shall start developing a broad abstract framework, in which the previous example of $\mathbb{R}^n$ as well as linear transform. $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be treated as basic examples.

Def: A vector space is a nonempty set V of objects, called vectors, on which two operations are defined: addition and multiplication by scalars (real numbers or more generally any field) subject to the following axioms:

1) For any $u, v \in V$, their sum $u+v$ is also in V ← basically says that we have "addition"
2) For any $u \in V$ and scalar c, $c \cdot u$ is also in V ← basically says that we have "mult. by scalars"
3) $u+v = v+u$
4) $(u+v)+w = u+(v+w)$
5) there exists a zero vector 0 in V such that $u+0=u$
6) $c(u+v) = cu+cv$
7) $(c+d)u = cu+du$
8) $c(du) = (cd)u$
9) $1 \cdot u = u$, $0 \cdot u = 0$

Rmk: Textbook also mentions

10) For any u in V, there is $-u$ in V such that $u+(-u)=0$,

BUT this is a consequence of 7) & 9) above.

Examples: $\mathbb{R}^n$, $\mathrm{Mat}_{m\times n}$ ($m\times n$ matrices), $C(\mathbb{R})$ (all continuous functions on $\mathbb{R}$)

$\mathbb{P}_n$ (the space of degree $\leq n$ polynomials)

$\mathbb{S} = \{(\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \mid y_k \in \mathbb{R}\}$ – the space of doubly infinite sequences of numbers

Important: In the above examples of $\mathbb{P}_n$ or $C(\mathbb{R})$, we treat rather complicated objects (functions) merely as single elements (vectors) of the corresponding vector spaces

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Def: A subspace of a vector space V is a subset H of V satisfying
1) $0 \in H$
2) If $u, v \in H$, then $u+v \in H$
3) If $u \in H$, then $c \cdot u \in H$ for any scalar c

Note: H becomes a vector space as all axioms from p. 1 obviously hold.

Examples:
i) $\{0\} \in V$ is the subspace, called zero subspace.
ii) $\mathbb{P}_n$ may be viewed as a subspace of $C(\mathbb{R})$
iii) $\{f \in C(\mathbb{R}) \mid f(0)=0\}$ – subspace of $C(\mathbb{R})$
iv) $\{f \in \mathbb{P}_n \mid \underbrace{f(x)=f(-x)}_{\text{even polynomials of degree } \le n}\}$ – subspace of $\mathbb{P}_n$
v) $\mathbb{S}_f$ – subset of $\mathbb{S}$ consisting of those $(\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$ with only finitely many non-zero y_k — subspace of $\mathbb{S}$.
vi) The vector space $\mathbb{R}$ is not a subspace of $\mathbb{R}^2$, since it is not even a subspace. HOWEVER, if you consider $H = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid x \text{ – any scalar}\right\}$, then clearly H is a subspace of $\mathbb{R}^2$ which looks exactly as $\mathbb{R}$.

Note: the line in $\mathbb{R}^2$ not containing the origin is NOT a subspace of $\mathbb{R}^2$.

vii) $\underbrace{\{A \in Mat_{n\times n} \mid A = A^T\}}_{\text{symmetric matrices}}$ – subspace of $Mat_{n\times n}$
viii) $\underbrace{\{A \in Mat_{n\times n} \mid A = -A^T\}}_{\text{skew-symmetric matrices}}$ – subspace of $Mat_{n\times n}$

Note: $\{A \in Mat_{n\times n} \mid A^2=0\}$ is NOT a subspace.

Q: Why?

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Given a set $S=\{\vec{v}_1, \ldots, \vec{v}_k\}$ of vectors of a vector space V, one defines the notions of linear combinations as well as the span $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ exactly as before.

CLAIM: If $\vec{v}_1, \ldots, \vec{v}_k$ are elements of a vector space V, then $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a subspace of V.

Discuss why it is so.

Terminology: $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ - the subspace of V spanned/generated by $\vec{v}_1, \ldots, \vec{v}_k$

For any subspace H of V, a spanning/generating set for H is a set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in H such that $H=\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

Ex1: Find spanning sets of:

1) The subspace of skew-symmetric 3x3 matrices in $Mat_{3\times 3}$
2) The subspace of symmetric 3x3 matrices in $Mat_{3\times 3}$
3) The subspace of even polynomials in $\mathbb{P}_6$
4) The subspace $\left\{\begin{pmatrix} a-b \\ a+b \\ 2b \end{pmatrix}\right\}$ in $\mathbb{R}^3$

1) $$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a\cdot\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\cdot\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

spanning set

2) $$\begin{pmatrix} d & a & b \\ a & e & c \\ b & c & f \end{pmatrix} = a\cdot\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + b\cdot\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} + c\cdot\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + d\cdot\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

spanning set

3) $\{1, x^2, x^4, x^6\}$ - spanning set

4) $$\begin{pmatrix} a-b \\ a+b \\ 2b \end{pmatrix} = a\cdot\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + b\cdot\underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}$$

spanning set

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• §4.2 Null spaces, Column spaces, Row spaces, Linear Transformations

Def: The null space of an $m\times n$ matrix A, denoted $\text{Nul } A$, is the set of all solutions of $A\vec{x}=0$, i.e.

$$\boxed{\text{Nul } A = \{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=0\}}$$

We already saw this concept back in §2.8!

The following is almost obvious:

CLAIM: $\text{Nul } A$ is a subspace of $\mathbb{R}^n$ ← Q: Ask if it is clear to everyone.

Recall: algorithm for producing a spanning set of $\text{Nul } A$.

Def: The column space of an $m\times n$ matrix A, denoted $\text{Col } A$, is the set of all linear combinations of the columns of A, i.e.

$$\boxed{\text{Col } A = \text{Span}\{\vec{a}_1, \ldots \vec{a}_n\} \text{ if } A = \begin{pmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{pmatrix}}$$

Again, this concept was already discussed back in §2.8.

Note: $\text{Col } A = \{\vec{b}\in\mathbb{R}^m \mid \vec{b}=A\vec{x} \text{ for some } \vec{x}\in\mathbb{R}^n\}$, hence,

$\text{Col } A$ = range of the corresponding linear transformation

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longmapsto A\vec{x}.$$

Rem: $\underbrace{\dim(\text{Col } A)+\dim(\text{Nul } A)}_{=\ \#\text{ columns of } A}$ as established in §2.9

The following is obvious:

CLAIM: $\text{Col } A$ is a subspace of $\mathbb{R}^m$

Def: The row space of an $m\times n$ matrix A, denoted $\text{Row } A$, is the set of all linear combinations of the rows of A, i.e.

$$\boxed{\text{Row } A = \text{Span}\{\vec{b}_1, \ldots \vec{b}_m\} \text{ if } A = \begin{pmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_m \end{pmatrix}}$$

CLAIM: $\text{Row } A$ is a subspace of $\mathbb{R}^n$

Q: Any relation b/w $\text{Row } A$ and $\text{Col } A^T$?
A: Coincide

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While Col A & Nul A provide the simplest constructions of subspaces of $\mathbb{R}^n$, it's natural to ask how they generalize once we replace $\mathbb{R}^n$ with more general vector spaces.

Def: A *linear transformation* T from a vector space V to a vector space W, denoted $T: V \to W$, is a rule that assigns to each $\vec{v} \in V$ a unique vector $T(\vec{v}) \in W$ so that

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in V$
2) $T(c\vec{u}) = c \cdot T(\vec{u})$ for any $\vec{u} \in V$ and scalar c

When $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, this coincides with the old definition we had.

The *kernel* (a.k.a. *null space*) of T is $\{\vec{v} \in V \mid T(\vec{v}) = 0\}$

CLAIM: Kernel of T is a subspace of V

The *range* of T is $\{\vec{w} \in W \mid \vec{w} = T(\vec{v})$ for some $\vec{v} \in V\}$

CLAIM: Range of T is a subspace of W.

Example: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, T = linear transf. with the standard matrix A.
Then: Kernel of T = Nul A
Range of T = Col A.

Ex 2: Consider $T: \text{Mat}_{n\times n} \longrightarrow \text{Mat}_{n\times n}$.

$$A \longmapsto A + A^T$$

Verify that T is a linear transformation. Describe kernel & range of T.

Kernel of T = skew-symmetric $n\times n$ matrices

Range of T = symmetric $n\times n$ matrices