Lecture #14

10/08/2020

- Last time: Vector Spaces
  Vector Subspaces of vector spaces
  Linear Combinations, Span, Spanning set

Ex1: Consider the set of all linear transformations $T: \mathbb{R}^n \to \mathbb{R}^m$.
Can you equip it naturally with a vector space structure?
What is the zero vector?

Ex2: Given any fixed matrices $X \in Mat_{m\times m}$ and $Y \in Mat_{n\times n}$, consider a subset $\{A \in Mat_{m\times n} \mid X \cdot A \cdot Y = 0\}$ of $Mat_{m\times n}$.
Is it a vector subspace of $Mat_{m\times n}$?

► Yes ▪ ← discuss

Ex3: a) Is a subset $\{A \in Mat_{m\times n} \mid$ columns are lin. independent$\}$ of $Mat_{m\times n}$, actually a vector subspace?

b) Is a subset $\{A \in Mat_{m\times n} \mid$ columns are lin. dependent$\}$ of $Mat_{m\times n}$, actually a vector subspace?

► a) No ← discuss
b) No ▪

- Discuss §4.2 following pages 4-5 of Lecture 13 Notes

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• §4.3 Linearly independent sets ; Bases

Similarly to the case of $\mathbb{R}^n$, we make the following definition

Def: An indexed set of vectors $\{\vec{v}_1, \ldots \vec{v}_k\}$ in a vector space $V$ is said to be linearly independent if the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

has only the trivial solution $c_1 = c_2 = \ldots = c_k = 0$

Otherwise, the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be linearly dependent

Similarly to the discussion for $\mathbb{R}^n$, we have

CLAIM: A set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ $(k \geq 2)$ with $\vec{v}_1 \neq \vec{0}$ is linearly dependent iff some $\vec{v}_j$ (with $1 < j \leq k$) is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

Q: When a set $\{\vec{v}_1\}$ is linearly dependent? ← only when $\vec{v}_1 = \vec{0}$.

Ex 4: a) Is the set $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ of $\mathrm{Mat}_{2\times 2}$ linearly independent?

b) Is the set $\{t^2-1, 2t+3, 5\}$ of $\mathbb{P}_2$ linearly independent?

c) Is the set $\{\cos t, \sin(t^2), 0, e^t\}$ of $C(\mathbb{R})$ linearly indep.?

a) No, as $(-2)\cdot\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + 1\cdot\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} + (-1)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$

b) Yes: if $\underbrace{a(t^2-1) + b(2t+3) + c\cdot 5}_{at^2 - a + 2b\cdot t + 3b + 5c} = 0$, then $a=0$, $b=0$, $5c - a + 3b = 0 \Rightarrow c = 0$

c) No: it contains $0$, e.g. $1\cdot 0 = 0$.

Def: Let $H$ be a subspace of a vector space $V$. A set of vectors $\mathcal{B}$ in $V$ is a basis for $H$ if

1) $\mathcal{B}$ is lin. indep. set

2) $H = \mathrm{Span}\,\mathcal{B}$, i.e. subspace spanned by $\mathcal{B}$ coincides with $H$.

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Ex 5: Verify that

a) $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}_n$.

b) $\{1, x^2, x^4, \ldots, x^{2\lfloor \frac{n}{2} \rfloor}\}$ is a basis for the subspace of "even polynomials" in $\mathbb{P}_n$.

c) $\left\{ E_{ij} = \begin{array}{c|c} & j^{th} \text{ column} \\ \hline i^{th} \text{ row} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{m \times n} \end{array} \right\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ — a basis for $Mat_{m \times n}$.

Ex 6: 1) Find a basis of the subspace of symmetric matrices in $Mat_{n \times n}$

2) Find a basis of the subspace of skew-symm. matrices in $Mat_{n \times n}$

Ex 7: Evoking Ex 4a), find a basis of $Span\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ in $Mat_{2\times 2}$.

As $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, any element in $Span\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ also belongs to $Span\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\right\}$.

On the other hand, we claim that the set $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\right\}$ is lin. indep, since if $a \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = 0 \Rightarrow \begin{cases} a + 2b = 0 \\ a + 3b = 0 \end{cases} \Rightarrow a = b = 0$.

So: $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\right\}$ — a basis of the above Span ∎

Note: We could also use the same argument to prove that $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ — also bases.

Based on the same ideas, one proves the following result:

CLAIM: Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space $V$. Let $H$ be the span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$, i.e. $H = Span\{\vec{v}_1, \ldots, \vec{v}_k\}$

a) If one of the vectors in $S$, say $\vec{v}_j$, is a linear combination of the remaining el-s in $S$, then the set formed from $S$ by removing $\vec{v}_j$ still spans $H$

b) If $H \neq \{\vec{0}\}$, some subset of $S$ is a basis for $H$

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Warning: Each nonzero subspace has infinitely many bases.
So, the previous claim just provides some of those!

Recall our previous discussions from §2.8, 2.9, where we explicitly described how to find a basis of Nul A, Col A for a given $m \times n$ matrix A.

Q: How to find a basis of Row A?

Clearly, the elementary row operations do not change Row A (unlike Col A), hence, we get:

CLAIM: A set of non-zero rows in an echelon form of A is a basis for Row A.