10/15/2020

Lecture #16

- Last time → Having a spanning set $\{\vec{v}_1, \dots, \vec{v}_n\}$ of a vector space V one can always get a basis by erasing a few terms.
  → Bases for Col A, Row A, Nul A.
  ! Make a remark about taking pivot columns of A for Col A, BUT pivot rows of echelon form for Row A
  → Dimension of V

Remark: Thus if V is spanned by $\vec{v}_1, \dots, \vec{v}_n$, we are assured to have $\dim V \leq n$.

Remark (Needed for homework): If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of V, then any vector $\vec{v} \in V$ can be uniquely written as $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$. The corresponding map $V \longrightarrow \mathbb{R}^n$, $\vec{v} \mapsto \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is 1-to-1, and as a result one can identify V with $\mathbb{R}^n$. Note that addition & scalar multiplication on V corresponds to those on $\mathbb{R}^n$.

- Discuss pp. 3-4 of Lecture 15 Notes.

Ex1: a) What is a maximal rank of a 4×6 matrix?
b) What is a maximal rank of a 6×4 matrix?

→ 4 in both cases ← discuss argument + provide examples ■

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- §5.1 Eigenvectors & eigenvalues

Ok, now back to practical computations, for today & next time we will focus on eigenvectors/ eigenvalues and their applications.

Def: An *eigenvector* of a square $n \times n$ matrix A is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ, called an *eigenvalue* of A.

Note: λ may be 0, BUT $\vec{v}$ is supposed to be nonzero.

Example: A vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$ with eigenvalue 2

as $\begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Q: But how can we find the eigenvectors & eigenvalues for a given matrix A?

Ex 2: Prove that $\lambda = 4$ is also an eigenvalue of the same $A = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$

Find the corresponding eigenvector.

4 is an eigenvalue iff $A\vec{v} = 4\vec{v}$ has a nontrivial solution.

Equivalently, the homogeneous eq-n $\underbrace{(A - 4I_2)}_{= \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}}\vec{v} = 0$ has a nontriv. sol-n.

The columns of $\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$ are clearly lin. dep., and the general solution has the form $x_3 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

So: We see that any nonzero multiple of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 4. ∎

As follows from this proof, we have:

Claim: λ - eigenvalue of $A \in Mat_{n \times n}$ iff $(A - \lambda I_n)\vec{v} = 0$ has a nontrivial solution.

We already discussed many different criterias to check that

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The set of all eigenvectors with eigenvalue λ is the same as the null-space of $A-\lambda I_n$.

‖ Def: This set is called the eigenspace of A corresponding to λ

Ex 3: Find all eigenvalues and corresponding eigenspaces of $A=\begin{pmatrix}5&3\\-1&1\end{pmatrix}$

▸ $\det\begin{pmatrix}5-\lambda&3\\-1&1-\lambda\end{pmatrix}=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4) \Rightarrow$ zero at $\lambda=2$, $\lambda=4$.

We saw that eigenspace of A corresponding to $\lambda=4$ is the line through $(-3,1)$

Likewise, the eigenspace of A corr. to $\lambda=2$ is the line through $(1,-1)$

Remark: If $\{\vec{v}_1,\ldots,\vec{v}_n\}$ is a basis of $\mathbb{R}^n$, which consists of eigenvectors of $A\in \mathrm{Mat}_{n\times n}$ (i.e. $A\vec{v}_i=\lambda_i\vec{v}_i$), then you can think of A as dilation by λ_i in the direction of $\vec{v}_i$.

Ex 4: Find an eigenspace of $A=\begin{pmatrix}4&2&3\\1&5&3\\1&2&6\end{pmatrix}$ corresponding to $\lambda=3$ and its basis.

▸ $A-3I_3=\begin{pmatrix}1&2&3\\1&2&3\\1&2&3\end{pmatrix} \xrightarrow[\text{reduct.}]{\text{row}} \begin{pmatrix}1&2&3\\0&0&0\\0&0&0\end{pmatrix}$

$$\left.\begin{array}{l}x_2, x_3 - \text{free}\\ x_1=-2x_2-3x_3\end{array}\right\} \Rightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-2x_2-3x_3\\x_2\\x_3\end{pmatrix}=x_2\begin{pmatrix}-2\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

So: $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$ – basis of that eigenspace

Rmk: It's always a good idea to verify that eigenvectors you found are correct, just by direct evaluation of $A\vec{v}$.

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Claim: The eigenvalues of a triangular matrix are the entries on its main diagonal

↑ Discuss why (if time permits)

Ex 5: Find eigenvalues and eigenspaces of
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

Claim: If $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors of A with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ – lin. indep. set

As we shall see soon, the ideal situation is when one can find a basis consisting of eigenvectors, BUT this is not always possible!