Lecture # 18

10/22/2020

- Last time → Characteristic equation, characteristic polynomial
  - → multiplicity of eigenvalues
  - → Similarity transformation $A \rightsquigarrow P^{-1}AP$
  - → Diagonalizable square matrices ($A = PDP^{-1}$, D-diagonal)

Ex 1: Is $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ diagonalizable?

▸ Char. polynomial of A is $(1-\lambda)^4 \Rightarrow$ only eigenvalue $\lambda = 1$ (multiplicity = 4)

If A was diagonalizable, i.e. $A = PDP^{-1}$ with D-diagonal, then char. pol-l of D should be the same, i.e. $(1-\lambda)^4$. But as D-diagonal, this implies all diagonal entries of D are 1.

Hence, $D = I_4 \Rightarrow PDP^{-1} = P \cdot I_4 \cdot P^{-1} = I_4 \neq A \Rightarrow$ Contradiction!

So: A is not diagonalizable. ◼

Last time, we also had the following two claims at the very end:

> **Claim**: An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors

> **Claim**: We have $A = PDP^{-1}$ with D-diagonal iff the columns of P are n linearly indep. eigenvectors of A, while the diagonal entries of D are the corresponding eigenvalues

Def: Given an $n \times n$ matrix A, a basis of $\mathbb{R}^n$ consisting of eigenvectors of A is called an *eigenvector basis of $\mathbb{R}^n$*.

Combining 1st claim with lin. indep. of eigenvectors for different eigenvalues, get:

> **Claim**: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Ex2: Diagonalize the matrix $A=\begin{pmatrix}2&1\\0&3\end{pmatrix}$, if possible.

Step 1: Find eigenvalues.

A - triangular $\Rightarrow$ eigenvalues are $\lambda=2,3$.

Step 2: Find lin. ind. eigenvectors

$\lambda=2$: $A-2I_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}: \begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}y\\0\end{pmatrix} \Rightarrow$ eigenvectors are $\begin{pmatrix}a\\0\end{pmatrix}$

Can take $\begin{pmatrix}1\\0\end{pmatrix}$.

$\lambda=3$: $A-3I_2=\begin{pmatrix}-1&1\\0&0\end{pmatrix}: \begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}-x+y\\0\end{pmatrix} \Rightarrow$ eigenvectors are $\begin{pmatrix}a\\a\end{pmatrix}$.

Can take $\begin{pmatrix}1\\1\end{pmatrix}$.

Step 3: Construct P from above eigenvectors.

$P=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

Step 4: Construct D from eigenvalues

$D=\begin{pmatrix}2&0\\0&3\end{pmatrix}$

So: $\begin{pmatrix}2&1\\0&3\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}2&0\\0&3\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$ $\leftarrow$ This was used in Ex7 in Lecture #17

Q: Can someone suggest an alternative proof of Ex1?

$\lambda=1$ - the only eigenvalue

$A-\lambda I_4=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}: \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\mapsto\begin{pmatrix}x_2\\0\\x_4\\0\end{pmatrix} \Rightarrow$ eigenspace $=\left\{\begin{pmatrix}*\\0\\*\\0\end{pmatrix}\right\}$ - 2-dim

$\Rightarrow$ can not pick an eigenbasis!

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Ex 3: Diagonalize the matrix $A=\begin{pmatrix}1&1&0&0\\0&2&0&0\\0&0&3&1\\0&0&0&4\end{pmatrix}$, if possible.

▶ Step 1: Find eigenvalues.

$\lambda = 1, 2, 3, 4.$

Step 2: Find eigenvectors.

$\lambda=1 \Rightarrow A-\lambda I_4=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&0&2&1\\0&0&0&3\end{pmatrix} : \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\longmapsto\begin{pmatrix}x_2\\x_2\\2x_3+x_4\\3x_4\end{pmatrix}$ - vanishes iff $x_2=x_3=x_4=0$

Can take $\vec{v_1}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$\lambda=2 \Rightarrow A-\lambda I_4=\begin{pmatrix}-1&1&0&0\\0&0&0&0\\0&0&1&1\\0&0&0&2\end{pmatrix} : \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\longmapsto\begin{pmatrix}-x_1+x_2\\0\\x_3+x_4\\2x_4\end{pmatrix}$ - vanishes iff $x_1=x_2$ & $x_3=x_4=0$

Can take $\vec{v_2}=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

$\lambda=3 \Rightarrow A-\lambda I_4=\begin{pmatrix}-2&1&&\\0&1&&\\&&0&1\\&&0&1\end{pmatrix} : \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\longmapsto\begin{pmatrix}-2x_1+x_2\\x_2\\x_4\\x_4\end{pmatrix}$ - vanishes iff $x_1=x_2=x_4=0$

Can take $\vec{v_3}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$\lambda=4 \Rightarrow A-\lambda I_4=\begin{pmatrix}-3&1&&\\0&-2&&\\&&-1&1\\&&0&0\end{pmatrix} : \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\longmapsto\begin{pmatrix}-3x_1+x_2\\-2x_2\\-x_3+x_4\\0\end{pmatrix}$ - vanishes iff $x_1=x_2=0$ & $x_3=x_4$

Can take $\vec{v_4}=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

Step 3: Construct P

$P=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$

Step 4: Construct D

$D=\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{pmatrix}$ ∎

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Let us conclude this discussion with the following criteria when a square matrix with less than n distinct eigenvalues is diagonalizable:

Claim: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_k$.

1) The dimension of the eigenspace for λ_j $(1 \le j \le k)$ is at most the multiplicity of the eigenvalue λ_j.

2) A is diagonalizable iff
   - the characteristic polynomial factors into linear factors

     AND
   - the dimension of the eigenspace for each λ_j equals exactly the multiplicity of λ_j.

3) If 2) holds, and $\mathcal{B}_j$ is a basis for the eigenspace corresponding to λ_j for each $1 \le j \le k$, then the union of all $\mathcal{B}_j$ $(1 \le j \le k)$ forms an eigenvector basis for $\mathbb{R}^n$.

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• § 5.4 Eigenvectors and linear transformations

Let's generalize the above discussion, by replacing:

- $\mathbb{R}^n$ with a general vector space V
- $n \times n$ matrix A with a linear transformation $T: V \longrightarrow V$.

Def: Let V be a vector space. An eigenvector of a linear transf. $T: V \longrightarrow V$ is a nonzero $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \cdot \vec{v}$ for some scalar λ, called an eigenvalue of T

Examples: 1) $V = \underbrace{C^1(\mathbb{R})}_{\text{differentiable f-s}}$, $T: f(x) \longmapsto f'(x)$ - linear operator.

Then: $f(x) = e^{\lambda x}$ is an eigenvector with eigenvalue λ.

2) $V = \mathbb{S}$, $T: (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \longmapsto (\ldots, y_{-1}, y_0, y_1, y_2, \ldots)$

Then: $(\ldots, \frac{1}{\lambda^2}, \frac{1}{\lambda}, 1, \lambda, \lambda^2, \ldots) \in \mathbb{S}$ is an eigenvector with eigenvalue λ.

Ex 4: Find all eigenvalues of $T: \mathbb{P}_n \to \mathbb{P}_n$ defined via $p(x) \longmapsto p'(x)$.

▶ If $p(x)$ is a degree $k \geq 1$ polynomial, then $p'(x)$ is a degree $k-1$ pol-l. Hence, it may not happen that $p'(x) = \lambda \cdot p(x)$ unless $p(x)$ is a degree 0 polynomial, i.e. $p(x) = p_0$ - constant. Then $p'(x) = 0 = 0 \cdot p(x)$.

So: The only eigenvalue is $\lambda = 0$.

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For the rest of today, we assume V to be fin. dimensional!

Choose a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V. Then, for any $\vec{x} \in V$, recall the coordinate vector $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^n$ defined as $\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$ with $\vec{x} = r_1\vec{b}_1 + \ldots + r_n\vec{b}_n$.

We also have $[T(\vec{x})]_{\mathcal{B}} \in \mathbb{R}^n$.

Q: What is the exact relation b/w $[\vec{x}]_{\mathcal{B}}$ and $[T(\vec{x})]_{\mathcal{B}}$?

If $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \Rightarrow \vec{x} = r_1\vec{b}_1 + \ldots + r_n\vec{b}_n \Rightarrow T(\vec{x}) = r_1T(\vec{b}_1) + \ldots + r_nT(\vec{b}_n)$

$$\Rightarrow [T(\vec{x})]_{\mathcal{B}} = r_1[T(\vec{b}_1)]_{\mathcal{B}} + \ldots + r_n[T(\vec{b}_n)]_{\mathcal{B}} = M \cdot [\vec{x}]_{\mathcal{B}},$$

where M is an $n \times n$ matrix given by $M = \begin{pmatrix} [T(\vec{b}_1)]_{\mathcal{B}} & \ldots & [T(\vec{b}_n)]_{\mathcal{B}} \end{pmatrix}$

matrix for T relative to the basis $\mathcal{B}$

also denoted $[T]_{\mathcal{B}}$.

Ex5: In the setup of Ex4, determine $[T]_{\mathcal{B}}$, where $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$.

$$[T]_{\mathcal{B}} = \underbrace{\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & 0 & \\ & & 0 & 3 & & \\ & & & 0 & \ddots & \\ & 0 & & & 0 & n \\ & & & & & 0 \end{pmatrix}}_{n+1} \Big\updownarrow n+1$$

In the particular case $V = \mathbb{R}^n$, we have a natural identification b/w $\{$linear transformations $\mathbb{R}^n \to \mathbb{R}^n\}$ and $\{n \times n$ matrices$\}$.

In particular, our previous discussion yields:

Claim: Suppose $A = PDP^{-1}$, P - invertible $n \times n$ matrix, D - diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed by the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\vec{x} \mapsto A\vec{x}$.

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More generally, if A & C are similar, i.e.

$$A = PCP^{-1}$$

then C is the $\mathcal{B}$-matrix for the transformation $\vec{x} \mapsto A\vec{x}$ when the basis $\mathcal{B}$ is formed by the columns of P.

! The converse is also true.

So, we get:

Claim: The set of all matrices similar to $A \in Mat_{n\times n}$ coincides with the set of all matrix representations of the linear transformation $\vec{x} \mapsto A\vec{x}$.

$\boxed{\text{Important Technical Remark}}$!!!

To compute $P^{-1}AP$, one does not need actually to evaluate P^{-1}!

Instead:

1) Compute $A \cdot P$
2) Row reduce $[P \,\vdots\, AP]$ to $[I_n \,\vdots\, P^{-1}AP]$