Lecture #19 10/27/2020

- Last time $\longrightarrow$ Diagonalize a square matrix, if possible (recall 4 steps!)
  - $\longrightarrow$ Criteria for when a square matrix is diagonalizable (ask if anyone remembers)
  - $\longrightarrow$ Eigenvalues of linear transformations $T: V \longrightarrow V$

Ex 1: 1) Verify that $p(t) \mapsto t \cdot \frac{\partial p}{\partial t}$ defines a linear transformation $T: \mathbb{P}_n \longrightarrow \mathbb{P}_n$

2) Find all eigenvalues and eigenspaces

3) In the basis $\{1, t, t^2, \ldots, t^n\}$, compute $[T]_{\mathcal{B}}$.

- Let's recall the construction of $[T]_{\mathcal{B}}$ for a chosen basis $\mathcal{B}$ of V. Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis of V. For any $\vec{x} \in V$, consider the coordinate vectors $[\vec{x}]_{\mathcal{B}}$ and $[T(\vec{x})]_{\mathcal{B}}$.

Fact: $\underbrace{[T(\vec{x})]_{\mathcal{B}}}_{\text{column of height } n} = \underbrace{[T]_{\mathcal{B}}}_{} \cdot \underbrace{[\vec{x}]_{\mathcal{B}}}_{\text{column of hight } n}$

"$\mathcal{B}$-matrix of lin. transf. T" — $n \times n$ matrix whose columns are $T(\vec{b}_1)_{\mathcal{B}}, \ldots, T(\vec{b}_n)_{\mathcal{B}}$

In the particular case $V = \mathbb{R}^n$, linear transform. $V \to V$ are in bijection with $n \times n$ matrices. Then, the above yields:

Claim: Suppose $A = PDP^{-1}$, P – invertible $n \times n$ matrix, D – diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed by the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\vec{x} \mapsto A\vec{x}$

Remark: If V is a fin. dimensional vector space, then choosing a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V allows to identify $V \xrightarrow{\sim} \mathbb{R}^n$, $\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$.

Then the moral of $T \rightsquigarrow [T]_{\mathcal{B}}$ is to identify linear transformations $V \longrightarrow V$ with $n \times n$ matrices. In that context, eigenvalues/eigenvectors boil to previous discussion

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• § Appendix B: "Complex Numbers"

Def: A complex number is a number written in the form
$$z = a + b\cdot i$$
where $a, b \in \mathbb{R}$ and i is a formal symbol satisfying $i^2 = -1$.

a – real part of z, denoted $\mathrm{Re}\, z$

b – imaginary part of z, denoted $\mathrm{Im}\, z$

So: $z = 0$ iff $a = b = 0$, equivalently, $z_1 = z_2$ iff $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$ & $\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$

Def: $\mathbb{C}$ – the set of all complex numbers

Rem: Any real number $a \in \mathbb{R}$ is considered as a special case via $a + 0\cdot i$.

Key: $\mathbb{C}$ is endowed with addition & multiplication:
$$\boxed{\begin{aligned}(a+bi) + (c+di) &= (a+c) + (b+d)i \\ (a+bi)(c+di) &= (ac-bd) + (ad+bc)i\end{aligned}}$$

Subtraction is also clear:
$$\boxed{(a+bi) - (c+di) = (a-c) + (b-d)i}$$

Def: The conjugate of $z = a+bi$ is $\bar{z} := a - bi$.

Note: If $z = a+bi$, then $z\cdot\bar{z} = (a+bi)(a-bi) = a^2 + b^2 \in \mathbb{R}_{\geq 0}$.

Def: The absolute value (a.k.a. modulus) of $z = a+bi$ is $|z| := \sqrt{a^2+b^2}$.

Properties:
1) $z = \bar{z}$ iff $z \in \mathbb{R}$ (i.e. $\mathrm{Im}\, z = 0$)
2) $\overline{z+w} = \bar{z} + \bar{w}$
3) $\overline{zw} = \bar{z}\cdot\bar{w}$
4) $z\cdot\bar{z} = |z|^2 \geq 0$
5) $|zw| = |z|\cdot|w|$
6) $|z+w| \leq |z| + |w|$

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If $z \in \mathbb{C}$ and $z \neq 0$, then z has a multiplicative inverse $1/z = z^{-1} := \frac{\bar{z}}{|z|^2}$,

i.e. $z \cdot z^{-1} = z^{-1} \cdot z = 1$.

This allows to divide $\underbrace{z/w}_{:= z \cdot \frac{1}{w}}$ for two complex numbers $z, w \in \mathbb{C}$ with $w \neq 0$

Ex 2: Let $z = 1+2i$, $w = 3+4i$.

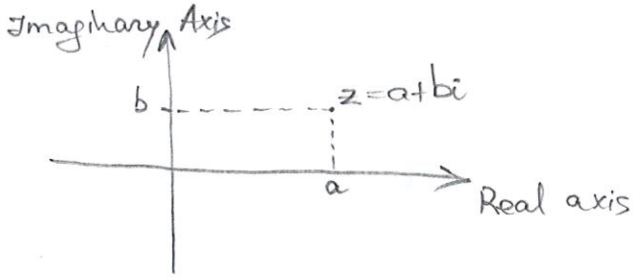
a) Compute $|z|$, $|w|$.

b) Compute $z+w$, $z \cdot w$.

c) Compute $\frac{z}{w}$.

Geometric Interpretation

Each $z = a+bi \in \mathbb{C}$ can be naturally depicted by the point (a,b)



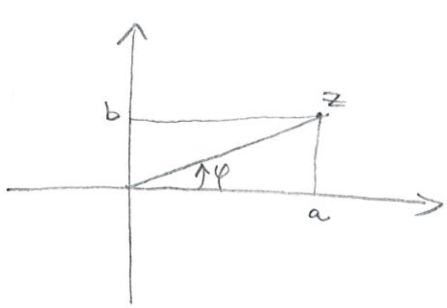
Q: What is the geometric meaning of $|z|$?

Q: What is the geometric meaning of the conjugate $\bar{z}$?

Q: What is the geometric meaning of the addition of two complex numbers?

To give a geometric representation of complex multiplication, one needs to use polar coordinates in $\mathbb{R}^2$.

Def: The argument of $z = a+bi$ is the angle φ between the positive real axis and the point (a,b) with conventions $-\pi < \varphi \leq \pi$.



Note: $\boxed{\begin{array}{l} a = |z| \cos\varphi \\ b = |z| \sin\varphi \end{array}} \Rightarrow \boxed{z = |z| (\cos\varphi + i \sin\varphi)}$

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Ex 3: If $z = |z|(\cos\varphi + i\sin\varphi)$, $w = |w|(\cos\theta + i\sin\theta)$, compute $z \cdot w$.

$$z \cdot w = |z| \cdot |w| \cdot \Big(\underbrace{(\cos\varphi\cos\theta - \sin\varphi\sin\theta)}_{\cos(\varphi+\theta)} + i \underbrace{(\cos\varphi \cdot \sin\theta + \sin\varphi \cdot \cos\theta)}_{\sin(\varphi+\theta)} \Big)$$

$$= |z| \cdot |w| \cdot \big(\cos(\varphi+\theta) + i \cdot \sin(\varphi+\theta) \big)$$

So: The product of two nonzero complex numbers is given in polar form by the product of their absolute values and the sum of their arguments

Q: What about the ratio of two nonzero complex numbers?

Note: Multiplication by $(\cos\varphi + i\sin\varphi)$ rotates $\circlearrowleft \varphi$.

Ex 4: a) Depict all complex numbers with $\operatorname{Re} z = 1$

b) —‖— with $|z| = 2$

c) —‖— with argument $= \pi$.

The above product formula allows to raise $z \in \mathbb{C}$ to any positive integer power:

if $z = r(\cos\varphi + i \cdot \sin\varphi)$, then $\boxed{z^k = r^k(\cos(k\varphi) + i \cdot \sin(k\varphi))}$

De Moivre's Theorem.

!!! While geometrically we identified $\mathbb{C}$ with $\mathbb{R}^2$, let me emphasize that $\mathbb{R}^2$ has only addition and scalar multiplication by $\mathbb{R}$, while $\mathbb{C}$ is naturally endowed with complex multiplication

Ex 5: Find all complex solutions of $z^5 = 2^5$