- Survey "Lecture Evaluation #2" ⟶ Please fill out by Saturday
- Last time ⟶ Complex numbers $\mathbb{C}$
  - ↳ addition, multiplication, subtraction
  - ⟶ conjugation ($\overline{a+bi} = a-bi$)
  - ⟶ division (using $(a+bi)^{-1} = \frac{a-bi}{a^2+b^2}$)
  - ⟶ geometric interpretation

We ended our Tuesday's class by proving:

> The product of two nonzero complex numbers is given in polar form by the product of their absolute values and the sum of their arguments

↑ If $\left.\begin{array}{l} z = |z|(\cos\varphi + i\sin\varphi) \\ w = |w|(\cos\theta + i\cdot\sin\theta) \end{array}\right\} \Rightarrow \boxed{zw = |z|\cdot|w|\cdot(\cos(\varphi+\theta) + i\sin(\varphi+\theta))}$

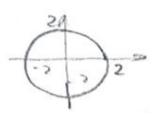
Similarly: If $w \neq 0$, then $\boxed{z/w = \frac{|z|}{|w|}(\cos(\varphi-\theta) + i\sin(\varphi-\theta))}$

Note: In particular, multiplication by $\cos\varphi + i\cdot\sin\varphi$ rotates by ↺ φ.

Ex 1: Depict all $z \in \mathbb{C}$ with

a) $\mathrm{Re}\, z = 1$ ⟵

b) $|z| = 2$ ⟵



c) (Argument of z) $= \pi$ ⟵

Ex 2: Verify $\overline{z+w} = \bar{z} + \bar{w}$ and $|z+w| \leq |z| + |w|$ using a geometric interpretation

Useful application of the above product formula is:

> De Moivre's Theorem: If $z = r(\cos\varphi + i\cdot\sin\varphi) \Rightarrow z^k = r^k(\cos(k\varphi) + i\sin(k\varphi))$

Ex 3: Find all complex solutions of $z^5 = 2^5$.

▸ Let $z = r(\cos\varphi + i\sin\varphi)$, then $z^5 = r^5\cdot(\cos(5\varphi) + i\sin(5\varphi))$.

So $z^5 = 2^5$ iff $r = 2$, $5\varphi = 2\pi k$ with $k \in \mathbb{Z}$

Hence, $z = 2\left(\cos\left(\frac{2\pi k}{5}\right) + i\cdot\sin\left(\frac{2\pi k}{5}\right)\right)$ with $k = 0, 1, 2, 3, 4$ (as k & $k+5$ yield same z) ∎

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§ 5.5 Complex Eigenvalues

The reason why we care about complex numbers is the following key result:

> **Claim:** Any polynomial $p(t)$ of degree n has exactly n roots, counting with multiplicities.

Example: For $p(t) = t^4 + 2t^2 + 1$, there are obviously no real roots as $p(t) > 0$ for $t \in \mathbb{R}$, but $p(t) = (t^2+1)^2 = (t-i)^2(t+i)^2 \Rightarrow$ it has complex roots i and $-i$, both with multiplicity 2 (and $2+2=4=\deg p(t)$).

In particular, we shall apply the previous eigenvalue/eigenvector theory for $\mathbb{C}^n$ (instead of $\mathbb{R}^n$), i.e. looking at $A\vec{x} = \lambda\vec{x}$ with $\lambda \in \mathbb{C}, \vec{x} \in \mathbb{C}^n$.

Basic Example: $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ with $0 < \varphi < \pi$.

$\det(A - \lambda I_2) = \lambda^2 - 2\cos\varphi \cdot \lambda + 1 \leftarrow$ it has roots $\lambda = \cos\varphi \pm i\sin\varphi$

For $\lambda = \cos\varphi + i\sin\varphi$: $A - \lambda I_2 = \begin{pmatrix} -i\sin\varphi & -\sin\varphi \\ \sin\varphi & -i\sin\varphi \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -i\sin\varphi \cdot x_1 - \sin\varphi \cdot x_2 \\ \sin\varphi \cdot x_1 - i\sin\varphi \cdot x_2 \end{pmatrix}$

$\Rightarrow$ e.g. $\vec{v_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ - eigenvector

$\lambda = \cos\varphi - i\sin\varphi$: $A - \lambda I_2 = \begin{pmatrix} i\sin\varphi & -\sin\varphi \\ \sin\varphi & i\sin\varphi \end{pmatrix} \Rightarrow$ e.g. $\vec{v_2} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ - eigenvector

So: Eigenvalues of A are $\cos\varphi + i\cdot\sin\varphi$ and $\cos\varphi - i\cdot\sin\varphi$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Note: Knowing exact values of eigenvalues, the above search of eigenvectors boils down to finding a solution of only one linear eq-n! (compare to discussion in the bottom of p. 305 of the book)

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- Similar to $\mathrm{Re}\, x$ and $\mathrm{Im}\, x$ for $x \in \mathbb{C}$, we defines $\mathrm{Re}\,\vec{x}, \mathrm{Im}\,\vec{x} \in \mathbb{R}^n$ for $\vec{x} \in \mathbb{C}^n$, so that $\boxed{\vec{x} = \mathrm{Re}\,\vec{x} + i \cdot \mathrm{Im}\,\vec{x}}$

  ($\mathrm{Re}\,\vec{x}$ — real part of $\vec{x}$; $\mathrm{Im}\,\vec{x}$ — imaginary part of $\vec{x}$.)

Example: $\mathrm{Re}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathrm{Im}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, so that $\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}$

- For $A \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ (i.e. $m\times n$ matrix with complex entries), define $\overline{A} \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ by taking complex conjugate of each entry of A.

Properties:
1) $\overline{r\vec{x}} = \overline{r} \cdot \overline{\vec{x}}$ for $r \in \mathbb{C}, \vec{x} \in \mathbb{C}^n$
2) $\overline{B\vec{x}} = \overline{B} \cdot \overline{\vec{x}}$ for $B \in \mathrm{Mat}_{m\times n}(\mathbb{C}), \vec{x} \in \mathbb{C}^n$.
3) $\overline{BC} = \overline{B} \cdot \overline{C}$ for $B \in \mathrm{Mat}_{m\times n}(\mathbb{C}), C \in \mathrm{Mat}_{n\times k}(\mathbb{C})$
4) $\overline{rB} = \overline{r} \cdot \overline{B}$ for $r \in \mathbb{C}, B \in \mathrm{Mat}_{m\times n}(\mathbb{C})$

In particular, if $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, then $\overline{A\vec{x}} = \overline{A} \cdot \overline{\vec{x}} = A \cdot \overline{\vec{x}}$

Hence: $\boxed{\text{If } A\vec{x} = \lambda\vec{x}, \text{ then } A\overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}}$

This implies:

> If A is a real square matrix, its complex eigenvalues occur in conjugate pairs
> (complex eigenvalues — i.e. those eigenvalues with $\mathrm{Im}\,\lambda \neq 0$)

Example: In the above example of $A = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$, we indeed got

$$\lambda_2 = \cos\varphi - i\sin\varphi = \overline{\cos\varphi + i\sin\varphi} = \overline{\lambda_1}$$

$$\vec{v_2} = \begin{pmatrix}1\\i\end{pmatrix} = \overline{\begin{pmatrix}1\\-i\end{pmatrix}} = \overline{\vec{v_1}}$$

Ex4: Find eigenvalues and eigenvectors for $A = \begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$

▸ $\lambda_1 = 1 + \sqrt{3}\cdot i$ $\quad \vec{v_1} = \begin{pmatrix}1\\-i\end{pmatrix}$

$\lambda_2 = 1 - \sqrt{3}\cdot i$ $\quad \vec{v_2} = \begin{pmatrix}1\\i\end{pmatrix}$

∎

> **Claim:** Let $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and the corresponding eigenvector $\vec{v} \in \mathbb{C}^2$. Then:
>
> $A = PCP^{-1}$, where $P = \begin{pmatrix}\mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v}\end{pmatrix}$ and $C = \begin{pmatrix}a & -b\\ b & a\end{pmatrix}$

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Ex 5 (Exercise 27 from p. 310): Let A be an $n\times n$ real matrix such that $A^T = A$
i.e. A - symmetric

Pick any $\vec{x} \in \mathbb{C}^n$ and set $q := \overline{\vec{x}}^T \cdot A \cdot \vec{x} \in \mathbb{C}$.

Show that q is a real number actually.

▶ $\bar{q} = \overline{\overline{\vec{x}}^T A \vec{x}} = \vec{x}^T \cdot \bar{A} \cdot \overline{\vec{x}} = \vec{x}^T \cdot A \cdot \overline{\vec{x}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \leftarrow$ obvious as we take transpose of a 1x1 matrix

$q = \overline{\vec{x}}^T \cdot A \cdot \vec{x} = \overline{\vec{x}}^T \cdot A^T \cdot \vec{x} = \left(\vec{x}^T \cdot A \cdot \overline{\vec{x}}\right)^T$ ■

§5.7 Applications to Differential Equations

Finally, we shall discuss a very important application of eigenvalue/eigenvector problem to differential eq-s.

Setup: Solving a system of diff. equations:

$$\begin{cases} x_1'(t) = a_{11}\cdot x_1(t) + \dots + a_{1n}\cdot x_n(t) \\ \vdots \\ x_n'(t) = a_{n1}\cdot x_1(t) + \dots + a_{nn}\cdot x_n(t) \end{cases}$$

a_{ij} – constants $(1 \le i,j \le n)$

$x_i(t)$ – differentiable functions

A nice way to rewrite this system is by using vector-valued function $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, so that we get

$$\boxed{\vec{x}'(t) = A \cdot \vec{x}(t)} \quad (*)$$

Clear: 1) $\vec{x}(t) = \vec{0}$ is a trivial solution

2) if $\vec{u}(t), \vec{v}(t)$ are solutions of $(*)$, so is any linear combination $c\cdot\vec{u}(t) + d\cdot\vec{v}(t)$.

General Result: The space of solutions of $(*)$ is n-dimensional

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<u>Simplest case</u>: A – diagonal, so that $A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$

Then the system is <u>decoupled</u> (i.e. each equation involves only 1 f-n):

$$\begin{cases} x_1'(t) = a_{11} \cdot x_1(t) \\ x_2'(t) = a_{22} \cdot x_2(t) \\ \vdots \\ x_n'(t) = a_{nn} \cdot x_n(t) \end{cases} \Rightarrow \begin{cases} x_1(t) = C_1 \cdot e^{a_{11} t} \\ x_2(t) = C_2 \cdot e^{a_{22} t} \\ \vdots \\ x_n(t) = C_n \cdot e^{a_{nn} \cdot t} \end{cases}$$

<u>So</u>: $\left\{ e^{a_{11} \cdot t} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e^{a_{22} \cdot t} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e^{a_{nn} \cdot t} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$ – basis of solutions

<u>Note</u>: Each of the above functions is of the form $\vec{v} \cdot e^{\lambda t}$, where λ is an eigenvalue of A (diagonal above) and $\vec{v}$ is the corresponding eigenvector.

More generally, if A is an $n \times n$ real matrix and $\vec{v}$ is an eigenvector with eigenvalue λ, i.e. $A\vec{v} = \lambda \vec{v}$, then

$$\boxed{\vec{x}(t) := e^{\lambda t} \cdot \vec{v} \text{ is a solution of } (*)}$$

<u>Indeed</u>: $\vec{x}'(t) = \lambda \cdot e^{\lambda t} \cdot \vec{v} = e^{\lambda t} \cdot \lambda \vec{v} = e^{\lambda t} \cdot A\vec{v} = A \cdot \vec{x}(t)$.

<u>Ex 6</u>: Solve $\vec{x}'(t) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}(t)$ with $\vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.