

- Last time → Inner product spaces
  - ↳ Recall 4 axioms
  - ↳ Recall two examples with  $V = \mathbb{P}_n$  or  $C[a, b]$ .
  - ↳ Can apply all results for  $\mathbb{R}^n$  from Section 6!
- Finish pages 5-6 of Lecture #26
- §7.1 Diagonalization of symmetric matrices

Recall that a matrix  $A$  is symmetric iff  $A^T = A$

(in particular, if  $A \in \text{Mat}_{m \times n} \Rightarrow A^T \in \text{Mat}_{n \times m} \Rightarrow m = n$ ).

Ex1: Let  $\text{Mat}_{n \times n}^{\text{sym}}$  denote the subset of  $\text{Mat}_{n \times n}$  consisting of symmetric ones

1) Verify that  $\text{Mat}_{n \times n}^{\text{sym}}$  is a subspace of  $\text{Mat}_{n \times n}$

2) Find a basis of  $\text{Mat}_{n \times n}^{\text{sym}}$

3) Compute  $\dim \text{Mat}_{n \times n}^{\text{sym}}$

Ex2: Diagonalize (if possible) symmetric matrix  $A = \begin{pmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{pmatrix}$

$$\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -6 & 4 \\ -6 & 2-\lambda & -2 \\ 4 & -2 & 3-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} + 6 \begin{vmatrix} -6 & -2 \\ 4 & 3-\lambda \end{vmatrix} + 4 \begin{vmatrix} -6 & 2-\lambda \\ 4 & -2 \end{vmatrix}$$

$$= (1-\lambda) \underbrace{(-10 + \lambda + \lambda^2)}_{-10 + \lambda + \lambda^2} + 6 \underbrace{(6\lambda + 26)}_{6\lambda + 26} + 4 \underbrace{(4 + 4\lambda)}_{4 + 4\lambda}$$

$$= -\lambda^3 + 63\lambda + 162 = -(\lambda^3 - 63\lambda - 162)$$

Note:  $-3$  is a root of this polynomial as  $-27 + 189 - 162 = 0$

$\Rightarrow \lambda + 3$  divides  $\lambda^3 - 63\lambda - 162$  and explicitly get:

$$\lambda^3 - 63\lambda - 162 = (\lambda + 3)(\lambda^2 - 3\lambda - 54) = (\lambda + 3)(\lambda + 6)(\lambda - 9)$$

So: eigenvalues are  $\lambda = -3, -6, 9$

## Lecture #27

### ▷ (Continuation of Ex 2)

As there are 3 distinct eigenvalues, each corresponding eigenspace is 1-dim and we need to choose nonzero v's in each.

$$\lambda = -3 \Rightarrow A + 3I_3 = \begin{pmatrix} 4 & -6 & 4 \\ -6 & 5 & -2 \\ 4 & -2 & 0 \end{pmatrix} \Rightarrow \text{choose } \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda = -6 \Rightarrow A + 6I_3 = \begin{pmatrix} 7 & -6 & 4 \\ -6 & 8 & -2 \\ 4 & -2 & 3 \end{pmatrix} \Rightarrow \text{choose } \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda = 9 \Rightarrow A - 9I_3 = \begin{pmatrix} -8 & -6 & 4 \\ -6 & -7 & -2 \\ 4 & -2 & -12 \end{pmatrix} \Rightarrow \text{choose } \vec{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

So:  $A = P \cdot D \cdot P^{-1}$  with  $P = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$  and  $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

BUT: the matrix  $P$  from the above proof has a nice property, namely, its columns form an orthogonal set (hence, an orthogonal basis) of  $\mathbb{R}^3$ . Indeed

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0.$$

From previous discussions we know that sometimes it's useful to have an orthonormal rather than orthogonal basis

So: choose  $\vec{u}_1 := \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ ,  $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$ ,  $\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$

Then:  $A = P \cdot D \cdot P^{-1}$  with  $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$  and  $P = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$   
||! P-orthogonal  
P · D · P<sup>T</sup>

## Lecture #27

Def: An  $n \times n$  matrix  $A$  is said to be orthogonally diagonalizable if there is a diagonal matrix  $D$  and an orthogonal matrix  $P$ :

$$A = P D P^{-1} = P D P^T$$

↑  
P-orthogonal  $\Leftrightarrow P^{-1} = P^T$

Rmk: A-orthogonally diagonalizable  $\Rightarrow$  A-diagonalizable.  
 $\Leftarrow$

Main Claim: An  $n \times n$  matrix  $A$  is orthogonally diagonalizable iff  $A$ -symmetric

Note: the direction " $\Rightarrow$ " is easy:

$$\text{if } A = P D P^T \Rightarrow A^T = (P D P^T)^T = (P^T)^T \cdot D^T \cdot P^T = P \cdot D \cdot P^T = A$$

The orthogonality of  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  from the proof of Ex 2 is not just a mere luck as the following result says:

Claim: If  $A$  is symmetric, then any two eigenvectors from different eigenspaces are orthogonal

↑ see p.420 for a proof



## Lecture #27

The above results may be summarized as follows:

Claim (Spectral theorem for symmetric matrices):

Let  $A$  be a symmetric  $n \times n$  matrix. Then:

- 1)  $A$  has  $n$  real eigenvalues, counting multiplicities
- 2) The dimension of the eigenspace for each eigenvalue  $\lambda$  equals the algebraic multiplicity of  $\lambda$
- 3) The eigenspaces are mutually orthogonal
- 4)  $A$  is orthogonally diagonalizable.

Let's conclude by obtaining another formula for  $A$  as above.

Let  $A = PDP^{-1} = PDP^T$  with  $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ ,  $P = \begin{pmatrix} \vec{u}_1 & & \vec{u}_n \end{pmatrix}$   
orthogonal

So  $\{\vec{u}_1, \dots, \vec{u}_n\}$  - orthonormal basis of  $\mathbb{R}^n$ .

Then: 
$$A = \begin{pmatrix} \vec{u}_1 & & \vec{u}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{pmatrix}$$

$$A = \lambda_1 \cdot \vec{u}_1 \cdot \vec{u}_1^T + \dots + \lambda_n \cdot \vec{u}_n \cdot \vec{u}_n^T \quad \leftarrow \text{spectral decomposition of } A$$

Here:  $\vec{u}_i$  is an  $n \times 1$  matrix,  $\vec{u}_i^T$  - an  $1 \times n$  matrix  $\Rightarrow \vec{u}_i \cdot \vec{u}_i^T$  - an  $n \times n$  matrix

Q: What is the rank of  $\vec{u}_i \vec{u}_i^T$ ?

A:  $\text{rk} = 1$ .