Lecture #29 12/03/2020

Revision Section #2

- Eigenvectors & Eigenvalues
  - → Definition and the case of triangular matrices
  - → Linear independence of eigenvectors corresp. to different eigenvalues
  - → Finding eigenvalues as roots of the characteristic polynomial $p(\lambda) := \det(A - \lambda \cdot I)$
  - → Finding eigenspace (& its basis) via solving the corresponding homogeneous linear system
  - → Similar matrices ($A \mapsto P^{-1}AP$ - similarity transformation)
  - → Diagonalization problem: writing A as $A = P \cdot D \cdot P^{-1}$, where P - invertible, D - diagonal
    - → Find eigenvalues $\rightsquigarrow$ determines D
    - → Find a basis of eigenspace $\rightsquigarrow$ determines P.
    - → Useful for many applications (in the class, discussed computation of A^k)
    - → Criteria for a diagonalization of A
  - → Generalization to linear transformations $T: V \to V$
    - → The matrix of a linear transf. w.r.t. a basis of V
  - → Complex eigenvalues
    - → complex numbers & operations on them
    - → for a real $n \times n$ matrix A, complex (non-real) eigenvalues come in conjugate pairs
    - → For a 2×2 real matrix A with a complex eig. $\lambda = a - bi$ $(b \neq 0)$ and associated eigenvector $\vec{v}$, we have a presentation $A = PCP^{-1}$, where $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $P = \left(\operatorname{Re}\vec{v} \;\; \operatorname{Im}\vec{v} \right)$
  - → Application to differential equations $\vec{x}'(t) = A \cdot \vec{x}(t)$ (repeller, attractor, saddle point, spiral point)
    - → Recall how to solve!

Lecture #29

- Inner Product
  Length
  Orthogonality

→ dot product in $\mathbb{R}^n$ and its properties

→ Length $\|\vec{v}\|$, Distance $dist(\vec{u},\vec{v})$, Angle b/w $\vec{u}$ & $\vec{v}$

→ Orthogonality

↳ Orthogonal complement of a subspace

↳ $(\text{Row } A)^\perp = \text{Nul } A$, $(\text{Col } A)^\perp = \text{Nul } A^T$

→ Orthogonal & orthonormal sets

↳ decomposition of any $\vec{y} \in W \subseteq \mathbb{R}^n$ w.r.t. an orthogonal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$ of W

$$\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \dots + \frac{\vec{y}\cdot\vec{u}_k}{\vec{u}_k\cdot\vec{u}_k}\vec{u}_k$$

↳ orthogonal projection of any $\vec{y} \in \mathbb{R}^n$ on W with an orthogonal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$

$\text{proj}_W \vec{y}$ = same formula as above!

→ Encoding vectors from an orthonormal set as a matrix A, get $A^T \cdot A = I$

↳ If A is square, then $A \cdot A^T = I$ as well

those A are called "orthogonal" →

→ Best Approximation Theorem:

$\hat{y} := \text{proj}_W \vec{y}$ is the point on W closest to $\vec{y}$

↳ equals $A \cdot A^T \cdot \vec{y}$ if columns of A form an orthonormal basis of W

→ Gram-Schmidt Process to construct orthogonal bases of W (recall what to do when we only have spanning set of W)

→ QR-factorization: $A = Q \cdot R$ ← triangular

↑ columns form an orthonormal set for Col A

↳ Q via Gram-Schmidt for Col A, R via $R = Q^T A$

→ Least square problems → $\hat{b} = \text{proj}_{\text{Col}A} \vec{b}$ – unique, $\hat{x}$ – not!

↳ Find $\hat{x}$ by solving $A^T \cdot A \cdot \vec{x} = A^T \cdot \vec{b}$

→ If columns of A – lin. ind, find $\hat{x}$ via $\hat{x} = R^{-1} \cdot Q^T \cdot \vec{b}$

special case when columns of A – orthogonal

Lecture #29

→ Inner Product Spaces
- → Definition & 4 axioms
- → Examples ($\mathbb{P}_n$, $C[a,b]$)
- → Generalization of length, distance, angles
- → Gram-Schmidt & Orthogonal projection
- → Best approximation
- → $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$ & $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

• Symmetric Matrices
($A = A^T$)
- → Basis & dimension
- → Notion of $A \in \mathrm{Mat}_{n \times n}$ being orthogonally diagonalizable
- → Main Result: A-orthog. diag $\iff$ A-symmetric

  $\Downarrow$

  Corollary: Any symmetric matrix is diagonalizable (over $\mathbb{R}$)
- → Spectral decomposition (p. 422)

  $$A = \lambda_1 \vec{u}_1 \cdot \vec{u}_1^T + \dots + \lambda_n \vec{u}_n \cdot \vec{u}_n^T$$

  where λ_i - eigenvalues

  $\vec{u}_i$ - orthonormal eigenvectors