Lecture #1

Linear Algebra MA 26500

- Office Hours : Tue & Thur, 10 - 11^{30} am, MATH 620
  or by appointment ($\to$ email me)

- Lectures $\to$ posted notes on Brightspace
  & homepage

- Hwk
  - $\nearrow$ Online – via MyLab Math
  - $\searrow$ Handwritten – via Gradescope (but grades will be in MyLab)

  $\downarrow$

  Each week you will have 3 online & 3 handwritten homework assignments (due 11^{59} pm next Monday)

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  Overall 36 online + 36 handwritten assignments
  (but we will drop 3 lowest scores in each category)

- PPHC $\nearrow$ if you get sick $\to$ get tested $\to$ if have to isolate still should get access to notes + Zoom Office Hours

  $\to$ If I get quarantined, then the lectures will go to Zoom mode for that time being

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§1.1 Systems of linear equations

a.k.a. unknowns

Def: A *linear equation* in the *variables* $x_1, \dots, x_n$ is an equation that can be written in the form

$$\boxed{a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b}$$

Here, b and the *coefficients* $a_1, \dots, a_n$ are usually given numbers

Examples: 1) $x - 5y = 2$ (as it is equivalent to $x_1 - 5x_2 = 2$)

Note: As we shall work with linear equations in 4, 5, ... variables we prefer to use (x_1, x_2) instead of (x, y).

Here: $a_1 = 1$, $a_2 = -5$, $b = 2$

2) $x_1 + 3x_2 + 10 = 0$ (as it is equivalent to $\underbrace{x_1 + 3x_2 = -10}$)

here: $a_1 = 1$, $a_2 = 3$, $b = -10$.

3) $x_1 + x_2 - 10x_3 = \pi \cdot x_1 + 1000$ (as it is equivalent to $\underbrace{(1-\pi)x_1 + x_2 - 10x_3 = 1000}$)

here: $a_1 = 1 - \pi$, $a_2 = 1$, $a_3 = -10$, $b = 1000$

Non-examples: 1) $x_1 + 2x_2 = x_3^2$

2) $x_1 + \sin(x_2) - 5x_3 = 2$

In this course, we should never worry about such complicated equations.

Def: A *linear system* (a.k.a. system of linear equations) is a collection of linear equations involving the same variables, say $x_1, x_2, \dots, x_n$.

Example:

$$x_1 - 3x_2 + x_3 = 3 \quad \longleftarrow \text{here: } a_1 = 1,\ a_2 = -3,\ a_3 = 1,\ b = 3$$

$$x_1 \qquad\quad - x_3 = 1 \quad \longleftarrow \text{here: } a_1 = 1,\ a_2 = 0,\ a_3 = -1,\ b = 1$$

Note: 1) It will be convenient for the future discussions to write 2nd equation in the above form, leaving out a blank spot for $0 \cdot x_2$.

2) You can also combine all equations involved by drawing $\{$ on the left of those.

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Let's start from a very simple highschool-level exercise:

Exercise #1: Find all solution of the system

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 - x_2 = 1 \end{cases}$$

Solution(s)

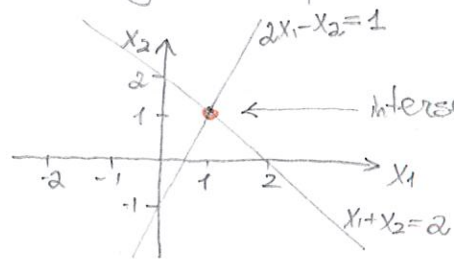
There are three different ways to proceed

Solution 1 (via substitution): $x_1 = 2 - x_2$ (from 1^{st} eq-n) $\xrightarrow{\text{plug into } 2^{nd} \text{ eq-n}}$ $\underbrace{2(2-x_2) - x_2 = 1}_{3-3x_2=0}$

$\Rightarrow \boxed{x_2 = 1 \Rightarrow x_1 = 1}$

Solution 2 (via elimination): Adding two equations, find $3x_1 = 3 \Rightarrow \boxed{x_1 = 1 \Rightarrow x_2 = 1}$

Solution 3 (by graphing):



$\leftarrow$ intersection point $(1,1) \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 1 \end{matrix}$

Note: From the last (geometric) approach it becomes clear that given two linear equations in two variables, they determine two lines, which are 1) intersect at 1 point, or 2) parallel, or 3) coincide.

And this is a very general feature!

CLAIM: Any linear system has 1) no solutions $\boxed{\text{OR}}$ 2) exactly one solution $\boxed{\text{OR}}$ 3) infinitely many solutions

$\leftarrow$ A solution of the linear system is an n-tuple $(s_1, s_2, \dots, s_n)$ of numbers such that each equation holds as you substitute $s_1, \dots, s_n$ for $x_1, \dots, x_n$.

Q: Which of those 3 approaches works better for linear systems in >2 variables?

A: ◦ Geometric approach is hard to visualize
(though for 3 variables, it boils down to intersection of planes in space)
◦ Substitution is ok, but becomes quite cumbersome.

So: we shall use the "elimination approach", but using some new notation.

Exercise #2: Solve (i.e. find all solutions) the following linear system:

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 4 \\ \qquad\quad x_2 - x_3 = 2 \end{cases}$$

As we shall be eliminating the variables one-by-one, all the essential information about the original linear system can be encoded by the 3×4 matrix (which is a rectangular array with 3 rows and 4 columns)

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & 2 \end{array}\right) \longleftarrow \text{called "augmented matrix" of the above system.}$$

Note: 1) In textbook, they use $[\ \dots\ ]$ and do not draw vertical line.
2) Erasing last column, get the coefficient matrix of the system.

Solution

- To eliminate variable x_1, we need to multiply 1^{st} eq-n by 2 and subtract from the 2^{nd} equation, which on the level of matrices looks as follows:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & 2 \end{array}\right) \xrightarrow{R_2 - 2\cdot R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

- Next, we switch the order of the 2^{nd} and 3^{d} equations, which on the level of matrices looks as follows:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & 1 & 2 \end{array}\right)$$

- Next, we eliminate x_2 from the 3^{d} equation, which is depicted via:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & 1 & 2 \end{array}\right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \end{array}\right)$$

Now, we are almost done, as we ended up with an equivalent linear system $\begin{cases} x_1 - x_2 + x_3 = 1 \\ \quad x_2 - x_3 = 2 \\ \qquad 4x_3 = -4 \end{cases}$

- From the 3^{d} eq-n get $x_3 = -1$
$\rightsquigarrow$ plug $x_3 = -1$ into 2^{nd} eq-n to get $x_2 = 1$
$\rightsquigarrow$ plug both in 1^{st} eq-n to get $x_1 = 3$.

So: The only solution is $(3, 1, -1)$, equivalently $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ ∎

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The above solution illustrates how "elimination procedure" works on the level of the corresponding augmented matrices. To be precise, one is applying so-called elementary row operations:

> 1) Replace a row by a sum of itself and a multiple of another row
> 2) Interchange two rows
> 3) Multiply all entries of a row by a nonzero constant

Note: All these operations are reversible!
As a result, we get

one is obtained from another via elementary row operations

> CLAIM: If the augmented matrices of two linear systems are row equivalent then the two systems have the same solution set.

Def: A linear system is called consistent if admits at least one solution.
A linear system is called inconsistent if it has no solution.

Exercise #3: Determine the value(s) of g such that the matrix $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & g & 10 \end{array}\right)$ is the augmented matrix of a consistent linear system

▸ $$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & g & 10 \end{array}\right) \xrightarrow{R_2-4R_1} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & g-8 & -2 \end{array}\right)$$

If $g \neq 8 \Rightarrow$ can recover x_2 uniquely $\left(x_2 = -\frac{2}{g-8}\right) \Rightarrow$ recover x_1

If $g = 8 \Rightarrow$ the 2^{nd} equation reads $0 = -2 \Rightarrow$ does not hold.

So: $g \neq 8$.

Exercise #4: Construct two different augmented matrices for linear systems with solutions $x_1 = 1$, $x_2 = 5$, $x_3 = -10$.

▸ E.g. the simplest one is $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -10 \end{array}\right)$

To obtain some other, apply elementary row operations, e.g. $R_2 - R_1$ gives:
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -10 \end{array}\right)$$

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§1.2 Row Reduction and Echelon Forms

Let's start from a couple of definitions, which apply to any matrix (not necessarily viewed as the augmented matrix)

Def: A rectangular matrix is in echelon form if:

1) All nonzero rows are above any rows of all zeros
2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
   (i.e. leftmost nonzero entry)
3) All entries in a column below a leading entry are zeros

Moreover, if it satisfies 4)-5) below, then a matrix is in reduced echelon form

4) The leading entry in each nonzero row is 1
5) Each leading 1 is the only nonzero entry in its column.

Def: A pivot is the same as the leading entry of a row, i.e. the leftmost nonzero entry of a row.

Example: The matrix $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & 4 \end{array}\right)$ we ended with in the solution of Exercise 2 is in echelon BUT not reduced echelon form.

However, performing elementary row operations to it:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \end{array}\right) \xrightarrow{\frac{1}{4}R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R_1-R_3]{R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

we end up with $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$ which is in the reduced echelon form.

> CLAIM: Each matrix is row equivalent to one and only one reduced echelon matrix