Lecture #2

- Last time: 1) Linear eq-n, linear system, solution of a linear system
  2) Elimination of variables to solve a linear system via elementary row operations, applied to the augmented matrix.
  (Recall them!)
  3) Notions of echelon AND reduced echelon forms

- Warming up

Ex1: For which values of numbers c, d the linear system

$$(*) \quad \begin{cases} x_1 + c x_2 = 3 \\ 2x_1 - 4x_2 = d \end{cases}$$

1) has exactly one solution
2) has infinitely many solutions
3) is inconsistent (i.e. has no solutions)

$$\left(\begin{array}{cc|c} 1 & c & 3 \\ 2 & -4 & d \end{array}\right) \xrightarrow{R_2 \mapsto R_2 - 2\cdot R_1} \left(\begin{array}{cc|c} 1 & c & 3 \\ 0 & -4-2c & d-6 \end{array}\right) \rightsquigarrow \text{corresponds to system} \begin{cases} x_1 + c x_2 = 3 \\ (-4-2c)x_2 = d-6 \end{cases}$$

Case 1: $-4-2c \neq 0$ (equivalently, $c \neq -2$)

Then, x_2 is uniquely recovered: $x_2 = \frac{d-6}{-4-2c}$. Plugging this value of x_2 into the 1st equation uniquely recovers x_1. Thus: there is exactly 1 solution.

Case 2: $-4-2c = 0$ (i.e. $c = -2$)

I) if $d-6 \neq 0$ (i.e. $d \neq 6$), then there is no x_2 such that $0 \cdot x_2 = d-6$
$\Rightarrow$ no solutions

II) if $d-6=0$ (i.e. $d=6$), then 2nd equation reads $0 \cdot x_2 = 0$ and any value of x_2 works. For any x_2, the 1st equation recovers x_1 uniquely. So: there are infinitely many solutions.

Conclusion: 1) $(*)$ has exactly one solution iff $c \neq -2$ & d-any.
2) $(*)$ has infinitely many solutions iff $c=-2$ & $d=6$
3) $(*)$ has no solutions iff $c=-2$ AND $d \neq 6$

Lecture #2

Before we proceed to the next example, let's recap 2 definitions:

Def: A *pivot position* in a matrix A is a location that corresponds to a leading entry 1 in the reduced echelon form.

Def: A *pivot column* is a column of A that contains a pivot position.

! Note: Pivot position and pivot column are already seen from an echelon form

e.g. in $\begin{pmatrix} \boxed{2} & 3 & -5 & 7 & 10 \\ 0 & \boxed{5} & 0 & 6 & 12 \\ 0 & 0 & 0 & \boxed{-1} & 3 \end{pmatrix}$ the boxed positions are pivot positions

pivot columns

Ex 2: Apply elementary row operations to transform

$$\begin{pmatrix} 0 & 1 & 2 & -3 & 1 & 5 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 1 & 2 & 7 & -2 & 0 & 17 \end{pmatrix}$$

first into an echelon form and then into the reduced echelon form.

$$\begin{pmatrix} 0 & 1 & 2 & -3 & 1 & 5 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 1 & 2 & 7 & -2 & 0 & 17 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 0 & 1 & 2 & -3 & 1 & 5 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & -3 & -6 & 9 & 0 & -18 \\ 0 & 1 & 2 & -3 & 1 & 5 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & 1 & 2 & -3 & 1 & 5 \\ 0 & -3 & -6 & 9 & 0 & -18 \end{pmatrix} \xrightarrow{R_3 \mapsto R_3 + 3R_1}$$

$$\rightsquigarrow \underbrace{\begin{pmatrix} \boxed{1} & 2 & 7 & -2 & 0 & 17 \\ 0 & \boxed{1} & 2 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 & \boxed{3} & -3 \end{pmatrix}}_{\substack{\text{already in echelon form} \\ \text{BUT not in reduced echelon}}} \xrightarrow{R_3 \mapsto \frac{1}{3}R_3} \begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & 1 & 2 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - R_3}$$

pivots

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 7 & -2 & 0 & 17 \\ 0 & 1 & 2 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1 \mapsto R_1 - 2R_2} \underbrace{\begin{pmatrix} \boxed{1} & 0 & 3 & 4 & 0 & 5 \\ 0 & \boxed{1} & 2 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{pmatrix}}_{\text{reduced echelon form!}}$$

Lecture #2

Let us now relate these row operations back to solving linear systems.

Ex 3 : Solve the following linear system

$$\begin{cases} x_2 + 2x_3 - 3x_4 + x_5 = 5 \\ 2x_1 + x_2 + 8x_3 + 5x_4 = 16 \\ x_1 + 2x_2 + 7x_3 - 2x_4 = 17 \end{cases}$$

▻ We start by writing down the corresponding augmented matrix:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & -3 & 1 & 5 \\ 2 & 1 & 8 & 5 & 0 & 16 \\ 1 & 2 & 7 & -2 & 0 & 17 \end{array}\right)$$

As this is exactly the matrix from Ex 2, we know by now that it is row equivalent to the following reduced echelon matrix:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 4 & 0 & 5 \\ 0 & 1 & 2 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Let us now write down the corresponding linear system:

$$\begin{cases} x_1 + 3x_3 + 4x_4 = 5 \\ x_2 + 2x_3 - 3x_4 = 6 \\ x_5 = -1 \end{cases}$$

The 3^d equation determines x_5 uniquely: $\boxed{x_5 = -1}$

But for any values of x_3 and x_4, the values of x_2 and x_1 are uniquely determined by 2^nd & 1^st equations:

$x_2 = 6 - 2x_3 + 3x_4$ AND $x_1 = 5 - 3x_3 - 4x_4$.

NOTE : x_3, x_4 – free, i.e. can take any values!

Answer :
$$\begin{cases} x_1 = 5 - 3x_3 - 4x_4 \\ x_2 = 6 - 2x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = -1 \end{cases}$$

Lecture #2

In the above examples, the variables x_1, x_2, x_5, corresponding to pivot columns are called "basic variables", while the other variables x_3, x_4 are called "free variables"

Rmk: a) The "free variables" can take any values, while "basic variables" will be uniquely determined by them.

b) Returning to the system
$$\begin{cases} x_1 + 3x_3 + 4x_4 = 5 \\ x_2 + 2x_3 - 3x_4 = 6 \\ x_5 = -1 \end{cases}$$
we could also let x_2, x_3 to take any values, then the 2^{nd} equation would uniquely determine x_4, and then the 1^{st} equation would uniquely determine x_1.

! However, it is customary to make "free variables" to take any values.

Rmk: While reducing a matrix to an echelon form we move from top rows to bottom rows, when transforming echelon to a reduced echelon form, we move backwards: from bottom rows to top rows (note: this is equivalent to substitution algorithm)

Q: What type of solutions does the system of Ex3 have?
(Answer: infinitely many)

Using the above argument, one arrives at the following result:

CLAIM: a) A linear system is consistent iff the rightmost column of the augmented matrix is NOT a pivot column, i.e. there is no row $(0 \dots 0 \mid b \neq 0)$ in its echelon form

b) If a linear system is consistent, then:
I) it has a unique solution iff there are no free variables
II) it has infinitely many solutions when there is at least one free variable.

Present / Discuss why this is so!

Lecture #2

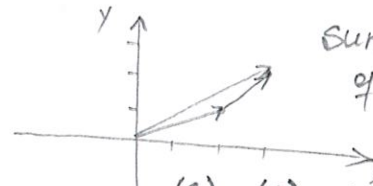
Let us now move to

§1.3 Vector equations

Def: A matrix with only one column is called a column vector (or just a vector)

Examples: $\begin{pmatrix}1\\3\end{pmatrix} \in \mathbb{R}^2$ ← 2-dimensional real plane

symbol "$\in$" means "belongs to".



summation of vectors

$$\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

Def: For any positive integer n, $\mathbb{R}^n$ denotes the collection of all $n\times 1$ column vectors.

There are two basic operations one can do with vectors:

1) Addition, e.g. $\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}+\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=\begin{pmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{pmatrix}$

2) Scalar Multiplication, e.g. $d\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}d\cdot a\\d\cdot b\\d\cdot c\end{pmatrix}$

Both are defined coordinate-wise.

Natural Properties (assume $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^n$ while $a,b\in\mathbb{R}$):

we use arrows above to indicate that those are vectors

$$\boxed{\begin{aligned}
&\vec{u}+\vec{v}=\vec{v}+\vec{u}\\
&(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})\\
&\vec{u}+\vec{0}=\vec{u}=\vec{0}+\vec{u}\\
&a(\vec{u}+\vec{v})=a\cdot\vec{u}+a\cdot\vec{v}\\
&(a+b)\vec{u}=a\vec{u}+b\vec{u}\\
&a(b\vec{u})=(ab)\vec{u}\\
&1\cdot\vec{u}=\vec{u}
\end{aligned}}$$

← here: $\vec{0}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ – so-called zero vector

Def: Given vectors $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_k\in\mathbb{R}^n$ and scalars $c_1,\dots,c_k\in\mathbb{R}$, we call $\vec{w}=c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_k\vec{v}_k$ a linear combination of $\vec{v}_1,\dots,\vec{v}_k$ (with weights $c_1,\dots,c_k$).

Def: Given $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_k\in\mathbb{R}^n$, the span of $\vec{v}_1,\dots,\vec{v}_k$, denoted $\mathrm{Span}\{\vec{v}_1,\dots,\vec{v}_k\}$ is a subset of $\mathbb{R}^n$ consisting of all linear combinations of $\vec{v}_1,\dots,\vec{v}_k$ (i.e. a subset of all $\vec{w}\in\mathbb{R}^n$ which can be written as $\vec{w}=c_1\vec{v}_1+\dots+c_k\vec{v}_k$)

Lecture #2

Q: Describe the span of

a) $\begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathbb{R}^3$

b) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \in \mathbb{R}^3$

c) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \in \mathbb{R}^3$

Let us conclude by relating vector equation

$$\boxed{x_1 \cdot \vec{v_1} + x_2 \cdot \vec{v_2} + \dots + x_k \cdot \vec{v_k} = \vec{w}}$$

to linear systems – whose augmented matrix has columns $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}, \vec{w}$.

Ex 4: Let $\vec{v_1} = \begin{pmatrix}1\\3\\-2\end{pmatrix}$, $\vec{v_2} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$, $\vec{w} = \begin{pmatrix}-4\\9\\8\end{pmatrix}$.

Determine if $\vec{w}$ belongs to Span$\{\vec{v_1}, \vec{v_2}\}$ or not.

This boils down to the question if there are scalars x_1, x_2 such that

$$x_1\begin{pmatrix}1\\3\\-2\end{pmatrix} + x_2\begin{pmatrix}-2\\1\\4\end{pmatrix} = \begin{pmatrix}-4\\9\\8\end{pmatrix}, \text{ equiv. solving } \begin{cases} x_1 - 2x_2 = -4 \\ 3x_1 + x_2 = 9 \\ -2x_1 + 4x_2 = 8 \end{cases}$$

Augmented matrix: $\left(\begin{array}{cc|c} 1 & -2 & -4 \\ 3 & 1 & 9 \\ -2 & 4 & 8 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 + 2R_1]{R_2 \mapsto R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 7 & 21 \\ 0 & 0 & 0 \end{array}\right)$

The latter encodes the linear system $\begin{cases} x_1 - 2x_2 = -4 \\ 7x_2 = 21 \\ 0 = 0 \end{cases} \Longrightarrow \begin{array}{c} x_2 = 3 \\ \Downarrow \\ x_1 = 2 \end{array}$

Answer: Yes ($\vec{w} = 2\vec{v_1} + 3\vec{v_2}$)