

Lecture #5

- Last time: 1) solutions sets of linear systems in parametric vector form
2) solutions sets of nonhomogeneous linear systems via solutions sets of the corresponding homogeneous systems
Recall the results/strategy, but no examples on that to save time

§1.7 Linear Independence

Recall that linear systems with all constants featuring in their right-hand sides are called homogeneous. In the matrix form, they can be equivalently written as $A\vec{x} = \vec{0}$

Example:
$$\begin{cases} x_1 + 4x_2 + 7x_3 = 0 \\ 2x_1 + 5x_2 + 8x_3 = 0 \\ 3x_1 + 6x_2 + 9x_3 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

As we saw last time, $\vec{x} = \vec{0}$ is always a solution of homog. systems. The question is if there are any others.

Today: rephrase this a bit.

Recall
$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

Hence: need to find weights x_1, x_2, x_3 so that the above linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ is a zero vector.

Hint: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \Rightarrow$ can take $x_1 = 1, x_2 = -2, x_3 = 1$.

And actually all solutions have the form $\vec{x} = t \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$
(Ask why?)

But this discussion leads us to the following important definition



Lecture #5

Def: An indexed set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in \mathbb{R}^n is said to be linearly independent if the vector equation

$$x_1\vec{v}_1 + \dots + x_k\vec{v}_k = \vec{0}$$

has only the trivial solution.

Otherwise, the set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be linearly dependent.

In the above example, we see that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \in \mathbb{R}^3$ are linearly dependent.

From what we started, the following result is tautological:

CLAIM: The columns of a matrix A are linearly independent iff the homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution.

Prmk: In practice, you will apply this result to determine if the given collection of vectors is linearly independent or not, by constructing matrix A whose columns are the given vectors.

Q: Is it true that $\{\vec{v}\}$ is linearly independent for any $\vec{v} \in \mathbb{R}^n$ set consisting of a single vector.

A: Yes, unless $\vec{v} = \vec{0}$ (as $x \cdot \vec{v} = \vec{0}$ with $x \neq 0$ iff $\vec{v} = \vec{0}$)

Q: Given any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, determine the criteria for $\{\vec{v}_1, \vec{v}_2\}$ to be linearly independent.

A: Neither of the two vectors is a multiple of another one.

The last question is closely related to the description of span of 2 vectors

CLAIM: 1) An indexed set $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ of two or more vectors is linearly dependent iff at least one of them is a linear combination of the others.

2) Furthermore, if S is linearly dependent and $\vec{v}_i \neq \vec{0}$, then for some $1 < j \leq k$, the vector \vec{v}_j is a linear combination of $\vec{v}_1, \dots, \vec{v}_{j-1}$.

Lecture #5

! Warning: the above claim does NOT say that each vector \vec{v}_j is a linear combination of the others - this is just NOT TRUE,

e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ - linearly dependent, BUT $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$

Note that if a set $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ contains the zero vector $\vec{0}$, then S is linearly dependent (as $1 \cdot \vec{0} = \vec{0}$).

There is one case, when one can assert that S is linearly dependent:

CLAIM: If $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in \mathbb{R}^n , and $k > n$, then S is linearly dependent

Reason: Boils down to the matrix equation $A\vec{x} = \vec{0}$, where A is an $n \times k$ matrix. As $k > n$, there is at least one free variable \Rightarrow there are nontrivial solutions.

EX 1: Is the set of vectors $\left\{ \begin{pmatrix} 100 \\ -314 \\ 2500 \end{pmatrix}, \begin{pmatrix} \pi \\ -e^2 \\ \sin(11) \end{pmatrix}, \begin{pmatrix} 7000 \\ -1025 \\ \cos(\pi) \end{pmatrix}, \begin{pmatrix} 1000000 \\ -\ln(105) \\ e^{30} \end{pmatrix} \right\}$

linearly dependent or independent?

As there are 4 vectors in \mathbb{R}^3 , $4 > 3$, this set is linearly dependent. But clearly finding a linear dependence relation is cumbersome.

EX 2: Determine if the following sets are linearly dependent or not:

a) $\left(\begin{pmatrix} 1 \\ -2 \\ 30 \\ 4 \end{pmatrix}, \begin{pmatrix} \pi \\ \sin(11) \\ e \\ \ln(23) \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ -90 \\ -12 \end{pmatrix} \right)$

b) $\left(\begin{pmatrix} 1 \\ -2 \\ 30 \\ 4 \end{pmatrix}, \begin{pmatrix} \pi \\ \sin(11) \\ e \\ \ln(23) \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ -90 \\ -13 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$

a) Lin. dependent as $3 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \vec{0}$ (equiv. $\vec{v}_3 = -3\vec{v}_1$)

b) Lin. dependent as this set contains $\vec{0}$

Rmk: Let's conclude with observation that if columns of an $n \times k$ matrix span \mathbb{R}^n , then each row has a pivot \Rightarrow # pivot columns = $n \Rightarrow n \leq k$

Lecture #5

Over the course of the previous weeks you have seen that linear systems can be equivalently stated as vector equation or matrix eq-n
 $x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{w}$ $A\vec{x} = \vec{b}$

And all 3 questions were treated by the same machinery: reducing the augmented matrix to a reduced echelon form by an iterative application of elementary row operations.

SO: we have not seen any real benefit of using vector or matrix equations.

But the matrix equation $A\vec{x} = \vec{b}$, or better to say its left-hand side, brings us to a very important topic:

§1.8 Linear Transformations

denoted $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Def: A transformation (a.k.a. function or mapping) T from \mathbb{R}^n to \mathbb{R}^m is a rule that assigns to each vector $\vec{x} \in \mathbb{R}^n$ a vector $T(\vec{x}) \in \mathbb{R}^m$

Here: \mathbb{R}^n - domain of T

\mathbb{R}^m - codomain of T

$T(\vec{x})$ - the image of \vec{x} under the action of T

$\text{Set } \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$ - the range of T .

For the rest of today, we are going to concentrate on those transformations T which arise via matrix multiplication.

In other words, fix an $m \times n$ matrix A . It defines a matrix transformation

$$\boxed{\mathbb{R}^n \ni \vec{x} \longmapsto A\vec{x} \in \mathbb{R}^m}$$

Q: What is the domain, codomain, and range of this transformation?

A: Domain = \mathbb{R}^n

Codomain = \mathbb{R}^m

Range = {all linear combinations of columns of A } = span {columns of A }

Lecture #5

Common simple questions (on matrix transformations $T(\vec{x}) = A\vec{x}$)

- 1) Compute image of $\vec{u} \in \mathbb{R}^n$: just evaluate $A \cdot \vec{u}$
- 2) Determine if \vec{w} is in the range of T : check the matrix equation $A\vec{x} = \vec{w}$ for consistency.
- 3) If $\vec{w} \in \mathbb{R}^m$ is in the range of T , find all $\vec{x} \in \mathbb{R}^n$ with image $= \vec{w}$: this boils down to a solution of the matrix equation $A\vec{x} = \vec{w}$ or equivalently the corresponding linear system.

Ex 3: Describe geometrically the following matrix transformations:

a) $\vec{x} \mapsto A\vec{x}$ with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 $\uparrow \mathbb{R}^2 \quad \uparrow \mathbb{R}^2$

b) $\vec{x} \mapsto A\vec{x}$ with $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
 $\uparrow \mathbb{R}^3 \quad \uparrow \mathbb{R}^3$

c) $\vec{x} \mapsto A\vec{x}$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
 $\uparrow \mathbb{R}^2 \quad \uparrow \mathbb{R}^2$

- a) Projects onto x-axis
b) Reflection in the xy-plane
c) 90° -rotation counterclockwise around the origin

Recall the two properties of the matrix-vector product:

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c \cdot \vec{u}) = c \cdot (A\vec{u})$$

Thus, if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$, then

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = c \cdot T(\vec{u})$$

← for any $\vec{u}, \vec{v} \in \mathbb{R}^n$
and any $c \in \mathbb{R}$

This brings us to the formal notion of linear transformations.

Lecture #5

We conclude with the following definition

Def: A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if

1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

2) $T(c \cdot \vec{u}) = c \cdot T(\vec{u})$ for any $\vec{u} \in \mathbb{R}^n, c \in \mathbb{R}$

True / False: $\{\vec{v}_1, \dots, \vec{v}_k\}$ - linearly independent
 \Downarrow (?)

$\{\vec{v}_k, \vec{v}_{k-1}, \dots, \vec{v}_2, \vec{v}_1\}$ - lin. independent

True / False: $\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k\}$ - linearly independent

\Downarrow (?)
 $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ - linearly independent.