

Lecture #6

Last time → Linear (in)dependence of a set S of vectors

Remember the general algorithm

- if $\vec{0}$ is in the set, then if this set is linearly dependent
- if S has more elements than their height \Rightarrow lin. dep.
- S-linearly indep. if the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution (A-matrix whose columns are all elements of S)

Ex 1: Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \in \mathbb{R}^4$ be a linearly independent set. Is it true that $\{\vec{v}_3, 2\vec{v}_2 + \vec{v}_1, \vec{v}_1 + \vec{v}_3\}$ is also lin. indep.

Assume
$$\underbrace{x_1(\vec{v}_3) + x_2(2\vec{v}_2 + \vec{v}_1) + x_3(\vec{v}_1 + \vec{v}_3)}_{(x_2 + x_3)\vec{v}_1 + 2x_2\vec{v}_2 + (x_1 + x_3)\vec{v}_3} = \vec{0}$$

But $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ being lin. independent

$$\Rightarrow \begin{cases} x_2 + x_3 = 0 \\ 2x_2 = 0 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 = 0 \end{cases}$$

So: $x_1 = x_2 = x_3 = 0$.

Hence: $\{\vec{v}_3, 2\vec{v}_2 + \vec{v}_1, \vec{v}_1 + \vec{v}_3\}$ - linearly independent

Transformations $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

→ matrix transformation $T(\vec{x}) = A\vec{x}$

Ex 2: a) A is a 3×10 matrix. What are the values of a, b to define a map $T: \mathbb{R}^a \rightarrow \mathbb{R}^b$ by $T(\vec{x}) = A\vec{x}$?

b) How many rows AND columns a matrix A must have in order to define a map $\mathbb{R}^n \rightarrow \mathbb{R}^5$?

a) $a = 10, b = 3$

b) 5 rows, 10 columns

Ex 3: Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be defined via $T(\vec{x}) = \begin{pmatrix} -30 & 0 & 0 & 0 \\ 0 & -30 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -30 \end{pmatrix} \vec{x}$

Compute the image of $\vec{x} = \begin{pmatrix} 1/2 \\ 1/3 \\ -1/5 \\ -1/10 \end{pmatrix}$

→ $T(\vec{x}) = A\vec{x} = \begin{pmatrix} -15 \\ -10 \\ 6 \\ 3 \end{pmatrix}$

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Recall the definition from the very end of last time

Def: A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if

- 1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$
- 2) $T(c\vec{u}) = c \cdot T(\vec{u})$ for any $\vec{u} \in \mathbb{R}^n, c \in \mathbb{R}$

Note that 1), 2) can be phrased as T preserving operations of vector addition and scalar multiplication.

Q: $T(\vec{0}) = ?$

(Pick any \vec{v} , then $\vec{0} = 0 \cdot \vec{v} \Rightarrow T(\vec{0}) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0}$)

Q: $T(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = ?$ (assuming T -linear)

($= T(\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) = T(\vec{v}_1) + T(\vec{v}_2 + \vec{v}_3) = \underline{T(\vec{v}_1) + T(\vec{v}_2) + T(\vec{v}_3)}$)

Q: $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k) = ?$ (assuming T -linear)

($= T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \dots + T(c_k\vec{v}_k) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \dots + c_kT(\vec{v}_k)$)

Ex 5: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation with $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}$
Compute $T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ 14 \end{pmatrix}$

Q: Is the transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ linear?

(A: No, e.g. $T(\vec{0}) \neq \vec{0}$
Also note that both 1) & 2) fail!)
 $(x_1, x_2, x_3) \mapsto (x_1 - 2x_2, 3x_3 + 2x_1 - 1, x_2 + x_3)$

Q: Can you determine $T\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ if $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear and

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}$

Q: How would you solve Ex 5 if you were given images

$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, T\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}$, and asked to find $T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$?

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§1.9 The matrix of a linear transformation

Evoking our solution of Ex 5 on the previous page immediately gives rise to the following general result:

CLAIM: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.

Then, there exists a unique $m \times n$ matrix A such that

$$T(\vec{x}) = A\vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

Explicitly, the matrix A is given by

$$A = \left(\begin{array}{ccc} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{array} \right) \leftarrow \text{i.e. its columns are images of } \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

the standard matrix for the linear transformation T .

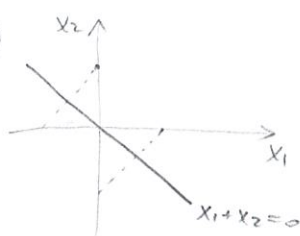
Reason: $T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = T \left(x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right) = x_1 T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 T \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n T \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = A\vec{x}$.

Ex 6: a) Find the standard matrix for $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $\vec{x} \mapsto -5\vec{x}$

b) Find the standard matrix for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which reflects through the line $x_1 + x_2 = 0$

a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} \Rightarrow A = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



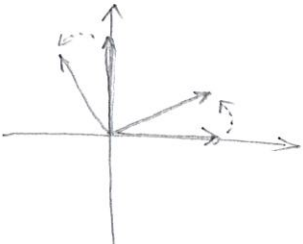
! So, the general recipe is to evaluate images of the standard vectors $\vec{e}_1, \vec{e}_2, \dots$, and combine them into the matrix!

Upshot: $\{\text{Linear transformations } \mathbb{R}^n \rightarrow \mathbb{R}^m\} = \{\text{Matrix transp. } \mathbb{R}^n \rightarrow \mathbb{R}^m\}$

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The following very important example is treated in the textbook (pp 76-77) but let's still do it right now:

Ex 7: Find the standard matrix of $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ - the rotation through the origin by angle φ counterclockwise.


$$\left. \begin{array}{l} (1) \mapsto (\cos \varphi) \\ (0) \mapsto (\sin \varphi) \\ (0) \mapsto (-\sin \varphi) \\ (1) \mapsto (\cos \varphi) \end{array} \right\} \Rightarrow A = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

Q: Back to the question from Tuesday class, what is the geometric meaning of the matrix transformation

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \vec{x} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$$

(A: rotation by $\pi/2$ counterclockwise)

! Look through examples on pages 78-80 in the textbook.

Ex 8: Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined via $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ 2x_2 - 7x_3 \\ x_3 - 5x_1 \end{pmatrix}$.

Show that it is matrix transformation by finding standard matrix.

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & -7 \\ -5 & 0 & 1 \end{pmatrix}$$

Def: 1) A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be onto \mathbb{R}^m if $\text{range}(T) = \mathbb{R}^m$, i.e. any $\vec{b} \in \mathbb{R}^m$ is an image of some $\vec{x} \in \mathbb{R}^n$.

2) A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be one-to-one if each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$.

Book: If the reduced echelon form of A has "free variables", then the transformation $\vec{x} \mapsto A\vec{x}$ is NOT one-to-one.

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Let's conclude with the following two general results:

CLAIM: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.

Then, T is one-to-one iff $T(\vec{x}) = \vec{0}$ has only the trivial solution.

CLAIM: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transp., A - the standard matrix.

1) T maps \mathbb{R}^n onto \mathbb{R}^m iff the columns of A span \mathbb{R}^m .

2) T is one-to-one iff the columns of A are linearly independent.

(See pages 81-82 of the textbook for proofs - if interested)

Ex: List all possible reduced echelon forms for 3×2 matrices with linearly dependent columns

Ex: List all possible reduced echelon forms for linear $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ that are onto.

Ex: If an 8×6 matrix A has linearly independent columns, then how many pivot columns does A have?