Lecture #7

Last time: $$\left\{\begin{array}{c}\text{Linear transformations}\\ T:\mathbb{R}^n\longrightarrow\mathbb{R}^m\end{array}\right\} = \left\{\begin{array}{c}\text{Matrix transformations}\\ \mathbb{R}^n \longrightarrow \mathbb{R}^m\\ \rotatebox{90}{$\in$} \qquad \rotatebox{90}{$\in$}\\ \vec{x} \longmapsto \underbrace{A}_{m\times n \text{ matrix}}\cdot\vec{x}\end{array}\right\}$$

In particular, we associated the standard matrix A to any linear T, explicitly given by:

$$A = \left(T\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \longrightarrow T\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}\right)$$

Q1: Provide an example of a linear map $\mathbb{R}^n \to \mathbb{R}^m$ which is:

a) onto and 1-to-1

b) onto but not 1-to-1

c) 1-to-1 but not onto

d) neither onto nor 1-to-1

Q2: a) Does there exist a 1-to-1 linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$

b) Does there exist an onto linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$

Conclusions from last week:

1) $\vec{x} \longmapsto A\vec{x}$ is 1-to-1 iff A has no free variables

2) $\overset{\text{linear}}{T}: \mathbb{R}^n \to \mathbb{R}^m$ is 1-to-1 iff $T(\vec{x}) = \vec{0}$ has only the trivial solution

3) linear $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto iff columns of A span $\mathbb{R}^m$ (A is the standard matrix of T)

4) linear $T: \mathbb{R}^n \to \mathbb{R}^m$ is 1-to-1 iff columns of A are linearly independent

Lecture #7

For the rest of today (to be continued next time), we shall discuss: various operations with matrices.

§2.1 Matrix Operations

First, given an $\underbrace{m\times n \cdot \text{ matrix}}_{m \text{ rows, } n \text{ columns}}$ A, we use a_{ij} $\left(\begin{smallmatrix}1\le i\le m\\ 1\le j\le n\end{smallmatrix}\right)$ to denote its entry in the i^{th} row & j^{th} column:

$$\boxed{A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \text{—} & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}\end{pmatrix}}$$

- Zero matrix 0: if all $a_{ij}=0$ (Note: For any m,n we have a zero matrix of size $m\times n$)
- Diagonal matrix: if $m=n$ and $a_{ij}=0$ for $i\neq j$, i.e. $A=\begin{pmatrix} * & & 0\\ & * & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$
- The identity matrix: if $m=n$ and $a_{ij}=\begin{cases}0, & \text{if } i\neq j\\ 1, & \text{else}\end{cases}$: $I_n=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$

Sum of matrices

If A, B are $m\times n$ matrices (they must be of the same size), then $A+B$ is the $m\times n$ matrix, whose $(i,j)^{th}$ entry is $\underbrace{\text{the sum of } (i,j)^{th} \text{ entries of } A,B}_{a_{ij}+b_{ij}}$

the "sum of A&B"

Scalar Multiple of matrices

Given an $m\times n$ matrix A and a scalar $c\in\mathbb{R}$, the scalar multiple cA is the $m\times n$ matrix whose columns are c times columns of A, i.e. $(i,j)^{th}$ entry of cA equals $c\cdot a_{ij}$

Q3: Let $A=\begin{pmatrix}1 & 2\\ -3 & 5\\ 7 & -4\end{pmatrix}$, $B=\begin{pmatrix}-2 & 1\\ 1 & 3\\ -1 & 4\end{pmatrix}$. Compute $2A-3B$

▶ $2A-3B=\begin{pmatrix}8 & 1\\ -9 & 1\\ 17 & -20\end{pmatrix}$ ∎

Properties:

A,B,C - of the same size

$c,d\in\mathbb{R}$ - scalars

1) $A+B=B+A$
2) $(A+B)+C=A+(B+C)$
3) $A+0=A=0+A$
4) $c\cdot(A+B)=c\cdot A+c\cdot B$
5) $(c+d)A=cA+dA$
6) $c(dA)=(cd)A$

Lecture #7

So Far, the addition and scalar multiplication were quite similar to the same named operations on $\mathbb{R}$. However, we'll soon discuss the very different features of the product and inverse of matrices (when they exist)

Q4: If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator with the standard matrix A

$T_2 : \mathbb{R}^n \to \mathbb{R}^m$ — // — B,

construct linear operator $\mathbb{R}^n \to \mathbb{R}^m$ which corresponds to:

a) $A+B$

b) $c \cdot A$

A : a) $T = T_1 + T_2$, i.e. $T(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x})$

b) $T = c \cdot T_1$, i.e. $T(\vec{x}) = c \cdot T_1(\vec{x})$

Which other operations can be applied to linear operators? We cannot multiply them clearly, but we can take compositions

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, dashed, bend right] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^k \end{tikzcd}$$

$\vec{x} \mapsto S(T(\vec{x}))$ ← clearly it is a linear operator. (check this!)

Hence, one can treat it as a matrix transformation.

$\boxed{Q}$ If the standard matrix of T is B (size $m \times n$) and the standard matrix of S is A (size $k \times m$), what is the standard matrix of their composition $(\vec{x} \mapsto S(T(\vec{x})))$?

$\boxed{A}$: If $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow T(\vec{x}) = B \cdot \vec{x} = \left(\vec{b_1} \dots \vec{b_n} \right) \cdot \vec{x} = x_1 \vec{b_1} + \dots + x_n \vec{b_n}$

$\Rightarrow S(\vec{x}) = S(x_1 \vec{b_1} + \_ + x_n \vec{b_n}) = x_1 S(\vec{b_1}) + \_ + x_n S(\vec{b_n}) = x_1 \cdot (A \cdot \vec{b_1}) + \dots + x_n (A \cdot \vec{b_n})$

and the latter can be written as

$$\underbrace{\left(A\vec{b_1} \dots A\vec{b_n} \right)} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

← the looked-after matrix

Lecture #7

This brings us to the following:

Def: If A is an $k \times m$ matrix, B is an $m \times n$ matrix, then the *product* AB is the $k \times n$ matrix whose columns are $A\vec{b}_1, \dots, A\vec{b}_n$:

$$\boxed{AB = \begin{pmatrix} A\vec{b}_1 & \dots & A\vec{b}_n \end{pmatrix}} \longleftarrow$$

note that this construction is much less obvious (not just component-wise)

Ex 1: Compute

a) $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot1+2\cdot2+3\cdot4 & 1\cdot(-2)+2\cdot1+3\cdot0 \\ -4\cdot1+0\cdot2+6\cdot4 & -4\cdot(-2)+0\cdot1+6\cdot0 \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 20 & 8 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & 6 \end{pmatrix}$

Note: For the product $A \cdot B$ to be well-defined, the number of columns in A must coincide with the number of rows in B.

Evoking the matrix-column rule for computation of $A\vec{b}$, we get

$$\boxed{(A \cdot B)_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \_\_ + a_{in} \cdot b_{nj}}$$

↑ *row-column rule* for computing $A \cdot B$

(reads: to find the $(i,j)^{th}$ entry of $A \cdot B$, take the sum of products of elements in the i^{th} row of A with corresponding elements in the j^{th} column of B.)

Ex2: Compute $\begin{pmatrix} 100 & 0 & 0 \\ 0 & -10 & 0 \\ e^2 & 204 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 19 & 17 & 134 \end{pmatrix} = 0 \leftarrow$ 3×3 ZERO MATRIX

Lecture #7

Properties of Matrix Multiplication

Let A be an $m\times n$ matrix and B, C be two matrices of appropriate size.

1) $A(BC) = (AB)C$
2) $A(B+C) = AB + AC$
3) $(B+C)A = BA + CA$
4) $r\cdot(AB) = (rA)\cdot B = A\cdot(rB)$ $\quad r\in\mathbb{R}$
5) $I_m \cdot A = A = A\cdot I_n$, where $I_k = \underbrace{\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}}_{k} \Big\updownarrow k$ – the identity matrix

Upshot: The addition and product of matrices satisfy the same rules as real number, in particular, can open parentheses in a usual way.

! **However**: $AB \neq BA$ even when both products are well-defined.

Ex 3: Compute AB and BA for

$$A = \begin{pmatrix}1 & 2\\ 3 & 5\end{pmatrix}, \quad B = \begin{pmatrix}-3 & 7\\ 1 & 6\end{pmatrix}$$

▶ $AB = \begin{pmatrix}-1 & 19\\ -4 & 51\end{pmatrix}$ $\quad BA = \begin{pmatrix}18 & 29\\ 19 & 32\end{pmatrix}$. Clearly $AB \neq BA$ ▪

! **More Warnings**:

1) If $AB = 0$, it is not true that $A = 0$ or $B = 0$, see Ex 2
2) If $AB = AC$, it is not true that $B = C$

Q5: For which matrices $A\cdot A$ makes sense?

Def: Given an $n\times n$ matrix A and positive integer $k > 0$, define *k-th power of A* as $\boxed{A^k := \underbrace{A\cdot A\cdot \ldots \cdot A}_{k \text{ times}}}$

Convention: $\boxed{A^0 := I_n}$

Q6 $A^k \cdot A^l = ?$ $\quad \nwarrow A^{k+l}$

Lecture #7

There is one more important operation on matrices that will come handy:

Def: Given an $m\times n$ matrix $A=(a_{ij})$, the transpose of A, denoted A^T, is the $n\times m$ matrix whose columns are formed from the corresponding rows of A. In other words:

$$\boxed{(A^T)_{ij} = a_{ji}}$$

example:

$$\begin{pmatrix}1\\3\\5\end{pmatrix}^T = (1\ 3\ 5)$$

$$\begin{pmatrix}1 & -3\\2 & -5\\7 & 10\end{pmatrix}^T = \begin{pmatrix}1 & 2 & 7\\-3 & -5 & 10\end{pmatrix}$$

Properties

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(c\cdot A)^T = c\cdot A^T \quad , c\in\mathbb{R}$
4) $(A\cdot B)^T = B^T\cdot A^T$

↑ WARNING: note the reversed order in the product on the right-hand side.

Ex 4: For $\vec{u} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\vec{v} = \begin{pmatrix}3\\2\end{pmatrix}$, evaluate $\vec{u}^T\cdot\vec{v}$ and $\vec{v}\cdot\vec{u}^T$

▶ $\vec{u}^T\cdot\vec{v} = (1\ 2)\begin{pmatrix}3\\2\end{pmatrix} = (7)$

$\vec{v}\cdot\vec{u}^T = \begin{pmatrix}3\\2\end{pmatrix}(1\ 2) = \begin{pmatrix}3 & 6\\2 & 4\end{pmatrix}$