Lecture #8

- Last time → Addition and multiplication by scalars for matrices
  → Multiplication of matrices
  → Transpose of a matrix (reverts rows ⟷ columns)

Q1: Is there a nonzero column vector $\vec{x} \in \mathbb{R}^n$ such that $\vec{x}^T \cdot \vec{x} = 0$? (A: No)

Q2: If A is a $k \times m$ matrix, and B is an $m \times 3$ matrix whose last column is the sum of the first two. What can be said about $A \cdot B$?
(A : $A \cdot B$ is a $k \times 3$ matrix whose last column is the sum of the first two)

§2.2 The inverse of a matrix

Def: An $n \times n$ matrix A is called invertible if there is another $n \times n$ matrix C such that:

$$\boxed{C \cdot A = \underbrace{I_n}_{\text{the } n\times n \text{ identity matrix}} = A \cdot C}$$

Q3: Can there be several matrices C satisfying these properties?
(A : NO. If $C_1 \cdot A = I_n = A \cdot C_2$, then:

$$C_1 = C_1 \cdot I_n = C_1 \cdot (AC_2) = (C_1 A) C_2 = I_n \cdot C_2 = C_2 \Rightarrow C_1 = C_2)$$

! Thus, if C exists, then it is unique and is denoted by $\boxed{\underbrace{A^{-1}}_{\text{the inverse of } A}}$

Ex1: Does the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$ admit an inverse?
(A : NO. If $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ was an inverse, we would have e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
But $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+3c & b+3d \\ 0 & 0 \end{pmatrix}$ has the (2,2)-entry equal to 0 NOT 1!)

Def: An $n \times n$ matrix A is called singular if it is not invertible.

For 2×2 matrices, this becomes very explicit, due to the following:

CLAIM: 1) The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff $ad - bc \neq 0$
2) If $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Def: The above quantity $\det(A) := ad - bc$ is called the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Q4: Compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$
(A : $A^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$)

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- Properties of inverse

1) A - invertible $\Rightarrow A^{-1}$ - invertible : $\boxed{(A^{-1})^{-1} = A}$

2) A, B - invertible $n\times n$ matrices $\Rightarrow AB$ - invertible : $\boxed{(AB)^{-1} = B^{-1}\cdot A^{-1}}$.

3) A - invertible $\Rightarrow A^T$ - invertible . $\boxed{(A^T)^{-1} = (A^{-1})^T}$

> **CLAIM** : If A is an invertible $n\times n$ matrix, then for any $\vec{b}\in\mathbb{R}^n$ there is a unique solution of $A\vec{x} = \vec{b}$, given explicitly by $\vec{x} = A^{-1}\cdot\vec{b}$

Question : How to compute A^{-1} (know the 2×2 case so far)?

> **CLAIM** : An $n\times n$ matrix A is invertible iff A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n, also transforms I_n to A^{-1}.

In order not to do the same work twice, the above claim can be interpreted as the following algorithm to find inverse A^{-1}:

> **Algorithm** : Row reduce the $n\times 2n$ matrix $(A \,\vdots\, I_n)$. If A is row equivalent to I_n, then $(A \,\vdots\, I_n)$ is row equivalent to $(I_n \,\vdots\, A^{-1})$. Otherwise, A is singular

Another perspective to this algorithm

The row reduction of $(A \,\vdots\, I_n)$ to $(I_n \,\vdots\, A^{-1})$ can be viewed as a simultaneous solution of n vector equations

$$A\vec{x} = \vec{e_1} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\quad A\vec{x} = \vec{e_2} = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \longrightarrow \quad A\vec{x} = \vec{e_n} = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

But by definition of the inverse and product, we exactly know that the i^{th} column of B is a solution of the i^{th} equation above.

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§2.3 Characterizations of invertible matrices

The notion of "invertible matrix" is closely related with previously discussed notions & questions due to the following important result:

> **CLAIM**: Let A be an $n\times n$ matrix. The following are equivalent:
> 1) A - invertible
> 2) A - row equivalent to I_n
> 3) A has n pivot positions
> 4) $A\vec{x}=\vec{0}$ has only the trivial solution
> 5) The columns of A form a linearly independent set
> 6) The linear transformation $\vec{x}\mapsto A\vec{x}$ is one-to-one
> 7) The linear transformation $\vec{x}\mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
> 8) The columns of A span $\mathbb{R}^n$.
> 9) The equation $A\vec{x}=\vec{b}$ has at least (and actually exactly) one solution for any $\vec{b}\in\mathbb{R}^n$.
> 10) There is an $n\times n$ matrix C such that $CA=I_n$
> 11) There is an $n\times n$ matrix D such that $AD=I_n$
> 12) A^T is invertible

Comment on 10) & 11): If $\left.\begin{array}{l} CA=I_n \overset{\text{CLAIM}}{\Rightarrow} A\text{-invertible} \Rightarrow C=A^{-1} \\ AD=I_n \overset{\text{CLAIM}}{\Rightarrow} D=A^{-1} \end{array}\right\} \Rightarrow$

> **CLAIM**: If A,B are $n\times n$ matrices satisfying $AB=I_n$, then A,B - invertible with $A^{-1}=B$ and $B^{-1}=A$

! Most efficient way (so far) is to apply 2) or 3) to decide if A is invertible or not.

Ex 2: Is the matrix $A=\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\13&14&15&17\end{pmatrix}$ invertible?

$A\sim\begin{pmatrix}1&2&3&4\\0&-4&-8&-12\\0&-8&-16&-24\\0&-12&-24&-35\end{pmatrix}\sim\begin{pmatrix}1&2&3&4\\0&1&2&3\\0&0&0&0\\ \cdots\end{pmatrix}\Rightarrow$ not invertible as we got zero row!

! The above Claim refers only to square matrices.

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Q: What is the meaning of this notion "invertible" in terms of the corresponding linear transformations $\mathbb{R}^n \to \mathbb{R}^n$?

First, let me give a definition:

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\boxed{T(S(\vec{x})) = \vec{x} = S(T(\vec{x})) \text{ for any } \vec{x} \in \mathbb{R}^n.} \quad (*)$$

CLAIM: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, A – its standard matrix. Then, T – invertible iff A – invertible matrix. In that case, the linear transformation $S: \vec{x} \mapsto A^{-1}\vec{x}$ is the unique solution S satisfying $(*)$

So: S is unique & linear (if it exists).

Ex 3: a) Is it true that a one-to-one linear transformation $\mathbb{R}^1 \to \mathbb{R}^3$ is onto?

b) Is it true that a one-to-one linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ is onto?

a) No, e.g. $T: x \mapsto \begin{pmatrix} x \\ x \\ x \end{pmatrix}$ is one-to-one, but $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ is not in the range.

b) Yes: by Claim on p.3: the standard matrix is invertible $\Rightarrow T$ is onto ∎

Ex 4: Assume $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(\vec{e_1}) = T(3\vec{e_2})$. Can T be onto?

$T(\vec{e_1}) = T(3\vec{e_2}) \Rightarrow T$ is not one-to-one $\Rightarrow$ its standard matrix A is not invertible

$\Downarrow$

T is not onto ∎

Let us now move to a new concept which is closely related with matrices and matrix equations.

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§ 2.8 Subspaces of $\mathbb{R}^n$

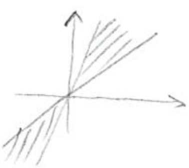
Def: A set H of $\mathbb{R}^n$ is called a *subspace of $\mathbb{R}^n$* if the following holds:

1) $\vec{0} \in H$
2) If $\vec{u}, \vec{v} \in H \Rightarrow \vec{u}+\vec{v} \in H$
3) If $\vec{u} \in H \Rightarrow c \cdot \vec{u} \in H$ for any $c \in \mathbb{R}$

Examples: $\{\vec{0}\}$, line $\{\mathbb{R} \cdot \vec{v}\}$, coordinate plane $\mathbb{R}^2_{x_1 x_2} \subseteq \mathbb{R}^3_{x_1 x_2 x_3}$

Counter examples:

or



or

Generalizing the above examples, we see that

Given any collection $\{\vec{v}_1, \_\_, \vec{v}_k\}$ of vectors in $\mathbb{R}^n$, their span $\text{Span}\{\vec{v}_1, \_\_, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$

! The converse is also true – shall be discussed below.

Let's now discuss two constructions producing subspaces given a matrix

Def: The *column space* of a matrix A, denoted *Col A*, is the set of all linear combinations of the columns of A (i.e. the span of columns)

For an $m \times n$ matrix $A \rightsquigarrow$ Col A is thus a subspace of $\mathbb{R}^m$
(! Col A coincides with the range of the corresponding linear transformation $\mathbb{R}^n \to \mathbb{R}^m$)

Def: The *null space* of a matrix A, denoted *Nul A*, is the set of all solutions of the homog. equation $A\vec{x} = \vec{0}$

Clearly, given an $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.

Note: While Col A is defined explicitly, Nul A is defined above quite implicitly. To get an explicit description of Nul A, one needs to write sol-ns of $A\vec{x} = \vec{0}$ in parametric form.

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Def: A *basis* of a subspace H of $\mathbb{R}^n$ is a linearly independent set in H which spans H.

Regarding the above two constructions, we have:

CLAIM 1: Solving homogeneous matrix equation $A\vec{x}=\vec{0}$ in the parametric vector form automatically yields a basis in $\text{Nul } A$.

CLAIM 2: The pivot columns of a matrix A form a basis of $\text{Col } A$

! Note that we say "a basis" as there are too many ways to pick one.

Ex 5: For $A=\begin{pmatrix} 1 & 2 & 7 & 10 & 12 \\ 2 & 5 & 8 & 11 & 6 \\ 4 & 6 & 9 & 5 & 0 \end{pmatrix}$ find a basis of $\text{Col } A$ and $\text{Nul } A$.