Lecture #11

• Last time → Determinants → original computation
→ reduction to triangular matrix via elementary row (or column) operations
→ $\det A = 0 \Leftrightarrow A$ - not invertible
→ $\det(A^T) = \det(A)$, $\det(AB) = \det(A)\det(B)$

Ex1: Let A be a 5×5 matrix with $\det(A) = 2$. Compute determinants of $3A$, A^3, $A^{-2}\cdot A^T$

$$\det(3A) = 3^5\cdot\det(A) = 486$$

$$\det(A^3) = \det(A)^3 = 8$$

$$\det(A^{-2}\cdot A^T) = \det(A)^{-2}\cdot\det(A^T) = \frac{1}{2}$$

Ex2: a) Compute $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

b) For which a, b, c, do $B = \left\{ \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix} \right\}$ form a basis of $\mathbb{R}^3$

c) — // — $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \right\}$ — // —

a) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} = (b-a)(c-a)(c-b)$$

b) If and only if $\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \neq 0 \overset{a)}{\Longleftrightarrow} a, b, c$ - pairwise distinct

c) Same answer as in b) as $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Recall: n vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in \mathbb{R}^n$ form a basis iff:

$$\det\left(\vec{v_1}\ \vec{v_2} \ldots \vec{v_n} \right) \neq 0.$$

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§3.3 Cramer's rule, Volume, and Linear Transformations

Now that we know the determinants, we can actually provide an explicit formula for solutions of the equation $\underline{A\vec{x}=\vec{b}}$, where A is an invertible square matrix (though in practice, it's very cumbersome to apply it for matrices A of big size)

CLAIM (Cramer's Rule): Let A be an invertible $n\times n$ matrix. For any $\vec{b}\in\mathbb{R}^n$, the unique solution of $A\vec{x}=\vec{b}$ has entries

$$\boxed{x_i = \frac{\det A_i(\vec{b})}{\det A}}, \quad 1\le i\le n,$$

where $A_i(\vec{b})$ is the $n\times n$ matrix obtained from A by replacing its i^{th} column with $\vec{b}$

Ex3: Use Cramer's rule to solve $\begin{cases} 2x_1-3x_2=-5 \\ 5x_1+2x_2=16 \end{cases}$

▶ $A=\begin{pmatrix} 2 & -3 \\ 5 & 2\end{pmatrix}$, $A_1(\vec{b})=\begin{pmatrix} -5 & -3 \\ 16 & 2\end{pmatrix}$, $A_2(\vec{b})=\begin{pmatrix} 2 & -5 \\ 5 & 16\end{pmatrix}$

$\det A = 19$ $\qquad \det A_1(\vec{b})=38 \qquad \det A_2(\vec{b})=57$

So: $x_1=\frac{38}{19}=2$, $\quad x_2=\frac{57}{19}=3$

Important Application: explicit formula for A^{-1}

Recall that the j^{th} column of A^{-1} is described as the solution of the equation $A\cdot\vec{x}=\vec{e}_j=\begin{pmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} \leftarrow j^{th}$ spot

Thus, applying Cramer's Rule, we get

$$\boxed{(i,j)^{th} \text{ entry of } A^{-1} \text{ equals } \frac{\det A_i(\vec{e}_j)}{\det A}}$$

But cofactor expansion down i^{th} column shows $\det A_i(\vec{e}_j)=(-1)^{i+j}\det A_{ji}=C_{ji}$ ← $(j,i)^{th}$ cofactor

A_{ji} obtained from A by deleting j^{th} row, i^{th} column

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All together we obtain:

CLAIM: Let A be an invertible $n\times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\cdot\begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \dots & C_{nn}\end{pmatrix}$$

where C_{ji} is the cofactor of A.

Def: The matrix $\begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & & & \\ \vdots & & \ddots & \\ C_{1n} & & & C_{nn}\end{pmatrix}$ is called the adjugate or adjoint of A, denoted $\operatorname{adj} A$

So: $\boxed{A^{-1} = \frac{1}{\det A}\cdot \operatorname{adj} A}$

! For 2×2 matrices A, this formula exactly coincides with previously dicussed $\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$.

Ex 4: Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0\end{pmatrix}$

$C_{11} = \begin{vmatrix} 1 & 5 \\ 6 & 0\end{vmatrix} = -30$, $C_{12} = -\begin{vmatrix} 0 & 5 \\ 5 & 0\end{vmatrix} = +25$, $C_{13} = \begin{vmatrix} 0 & 1 \\ 5 & 6\end{vmatrix} = -5$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 6 & 0\end{vmatrix} = +18$, $C_{22} = \begin{vmatrix} 1 & 3 \\ 5 & 0\end{vmatrix} = -15$, $C_{23} = -\begin{vmatrix} 1 & 2 \\ 5 & 6\end{vmatrix} = 4$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 5\end{vmatrix} = 7$, $C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5\end{vmatrix} = -5$, $C_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1\end{vmatrix} = 1$

$\det A = 1\cdot\begin{vmatrix} 1 & 5 \\ 6 & 0\end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ 1 & 5\end{vmatrix} = -30 + 5\cdot 7 = 5$

So: $A^{-1} = \frac{1}{5}\begin{pmatrix} -30 & 18 & 7 \\ 25 & -15 & -5 \\ -5 & 4 & 1\end{pmatrix} = \begin{pmatrix} -6 & \frac{18}{5} & \frac{7}{5} \\ 5 & -3 & -1 \\ -1 & 4/5 & 1/5\end{pmatrix}$

exactly coincides with Ex 1 from Lecture #9.

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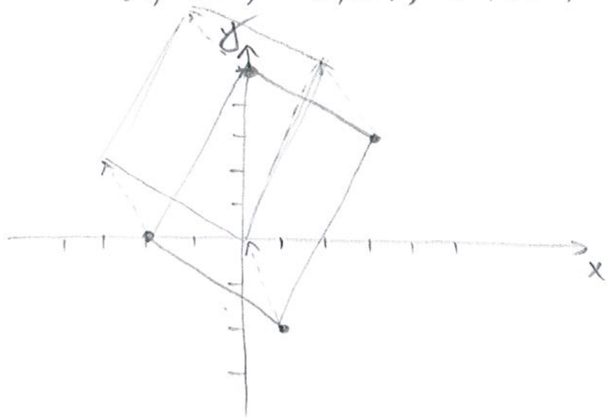
Determinants also have a very important geometric interpretation.

CLAIM: 1) If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A equals $\underbrace{|\det A|}_{\text{the absolute value of } \det A}$

2) If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A equals $|\det A|$.

Ex 5: Calculate the area of the parallelogram with 4 vertices: $(1,-2)$, $(3,3)$, $(0,5)$, and $(-2,0)$.



First we translate it to have the origin as one of the vertices. The new parallelogram has the same area and has the following vertices: $(0,0)$, $(2,5)$, $(-1,7)$, $(-3,2)$

Hence: Area $= \left|\det\begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix}\right| = \underline{\underline{19}}$

Another perspective to the above result is via the language of linear transformations:

CLAIM: 1) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{Area of } T(S)\} = \{\text{Area of } S\}\cdot|\det A|$$

2) Likewise, if $T:\mathbb{R}^3\to\mathbb{R}^3$ is determined by A, and S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{Volume of } T(S)\} = \{\text{Volume of } S\}\cdot|\det A|$$

Note: We recover the previous claim by taking S to be 1×1 stand. square or $1\times1\times1$ cube.

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Approximating any region in $\mathbb{R}^2$ (resp. in $\mathbb{R}^3$) by a union of tiny parallelograms (resp. parallelepipeds), we see

$$\boxed{\text{the above claim is valid for any region } S \text{ in } \mathbb{R}^2 \text{ or } \mathbb{R}^3, \text{ i.e. } \frac{\text{Area of } T(S)}{\text{Area of } S} = |\det A| \text{ or } \frac{\text{Volume of } T(S)}{\text{Volume of } S} = |\det A|}$$

Important illustration is provided in the following (see Example 5, p. 195 of textbook)

Ex6: For positive $a, b > 0$, find the area of the region E in $\mathbb{R}^2$ bounded by the ellipse $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

Key Observation: E is the image of the unit disk $D \subseteq \mathbb{R}^2$ under the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ determined by the matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

explain!

$\det A = ab$, $\text{Area}(D) = \pi \cdot 1^2 = \pi$.

So: $\boxed{\text{Area of } E = \pi . ab}$

Remark: Area of the triangle is $\frac{1}{2} \times$ Area of parallelogram.

Remark: Cramer's rule is useful when you need to compute a single x_i or need to find a single $(A^{-1})_{ij}$.