Lecture #13

• §4.1 Vector Spaces and Subspaces

Today we shall start developing a broad abstract framework, in which the previous example of $\mathbb{R}^n$ as well as linear transform. $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be treated as basic examples.

Def: (the key definition) A vector space is a nonempty set V of objects, called vectors, on which two operations are defined: addition and multiplication by scalars (real numbers or more generally any field) subject to the following axioms:

1) For any $\vec{u}, \vec{v} \in V$, their sum $\vec{u}+\vec{v}$ is also in V ← basically says that we have "addition"
2) For any $\vec{u} \in V$ and scalar c, $c \cdot \vec{u}$ is also in V ← basically says that we have "mult. by scalars"
3) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
4) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
5) there exists a zero vector $\vec{0}$ in V such that $\vec{u}+\vec{0}=\vec{u}$
6) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
7) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
8) $c(d\vec{u})=(cd)\vec{u}$
9) $1\cdot\vec{u}=\vec{u}$, $0\cdot\vec{u}=\vec{0}$

Rmk: Textbook also mentions

10) For any $\vec{u}$ in V, there is $-\vec{u}$ in V such that $u+(-u)=0$,

BUT this is a consequence of 7) & 9) above.

Examples: $\mathbb{R}^n$, $\mathrm{Mat}_{m\times n}$ ($m\times n$ matrices), $C(\mathbb{R})$ (all continuous functions on $\mathbb{R}$)

$\mathbb{P}_n$ (the space of degree $\leq n$ polynomials)

$\mathbb{S}=\{(\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \mid y_k \in \mathbb{R}\}$ – the space of doubly infinite sequences of numbers

Important: In the above examples of $\mathbb{P}_n$ or $C(\mathbb{R})$, we treat rather complicated objects (functions) merely as single elements (vectors) of the corresponding vector spaces

Warning: $\{\deg = n \text{ polynomials}\}$ do not form a vector space (for any $n>0$).

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Def: A subspace of a vector space V is a nonempty subset H of V satisfying

1) $\vec{0} \in H$ ← **Note**: 1) follows from 3) below
2) If $\vec{u}, \vec{v} \in H$, then $\vec{u}+\vec{v} \in H$ ← "H is closed w.r.t. addition"
3) If $\vec{u} \in H$, then $c \cdot \vec{u} \in H$ for any scalar c ← "H is closed w.r.t. scalar multiplic."

! **Note**: H becomes a vector space as all axioms from p. 1 obviously hold.

Examples:
i) $\{\vec{0}\} \in V$ is the subspace, called zero subspace.
ii) $\mathbb{P}_n$ may be viewed as a subspace of $C(\mathbb{R})$
iii) $\{f \in C(\mathbb{R}) \mid f(0)=0\}$ – subspace of $C(\mathbb{R})$

NOTE: But replacing $f(0)=1$ would lead to a NOT subspace

iv) $\{f \in \mathbb{P}_n \mid \underbrace{f(x)=f(-x)}_{\text{even polynomials of degree} \le n}\}$ – subspace of $\mathbb{P}_n$

v) S_f – subset of S consisting of those $(\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots)$ with only finitely many non-zero y_k — subspace of S.

vi) The vector space $\mathbb{R}$ is not a subspace of $\mathbb{R}^2$, since it is not even a subspace. HOWEVER, if you consider $H = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid x \text{ – any scalar}\}$, then clearly H is a subspace of $\mathbb{R}^2$ which looks exactly as $\mathbb{R}$.

Note: the line in $\mathbb{R}^2$ not containing the origin is NOT a subspace of $\mathbb{R}^2$.

vii) $\underbrace{\{A \in Mat_{n\times n} \mid A = A^T\}}_{\text{symmetric matrices}}$ – subspace of $Mat_{n\times n}$

viii) $\underbrace{\{A \in Mat_{n\times n} \mid A = -A^T\}}_{\text{skew-symmetric matrices}}$ – subspace of $Mat_{n\times n}$

Note: $\{A \in Mat_{n\times n} \mid A^2 = 0\}$ is NOT a subspace.

ix) $\{n\times n \text{ singular matrices}\}$ is NOT a subspace of $Mat_{n\times n}$
x) $\{n\times n \text{ invertible matrices}\}$ is NOT a subspace of $Mat_{n\times n}$

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Given a set $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ of vectors of a vector space V, one defines the notions of linear combinations as well as the span Span$\{\vec{v}_1, \ldots, \vec{v}_k\}$ exactly as before.

> **CLAIM:** If $\vec{v}_1, \ldots, \vec{v}_k$ are elements of a vector space V, then Span$\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a subspace of V.

Discuss why it is so.

Terminology: Span$\{\vec{v}_1, \ldots, \vec{v}_k\}$ – the subspace of V spanned/generated by $\vec{v}_1, \ldots, \vec{v}_k$

For any subspace H of V, a spanning/generating set for H is a set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in H such that $H = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

Ex1: Find spanning sets of:

1) The subspace of skew-symmetric 3×3 matrices in $Mat_{3\times3}$
2) The subspace of symmetric 3×3 matrices in $Mat_{3\times3}$
3) The subspace of even polynomials in $\mathbb{P}_6$
4) The subspace $\left\{\begin{pmatrix} a-b \\ a+b \\ 2b \end{pmatrix}\right\}$ in $\mathbb{R}^3$

1) $$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a\cdot\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\cdot\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

spanning set

2) $$\begin{pmatrix} d & a & b \\ a & e & c \\ b & c & f \end{pmatrix} = a\cdot\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + b\cdot\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} + c\cdot\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + d\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

spanning set

3) $\{1, x^2, x^4, x^6\}$ – spanning set

4) $$\begin{pmatrix} a-b \\ a+b \\ 2b \end{pmatrix} = a\cdot\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + b\cdot\underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}$$

spanning set