Lecture #14

Last time:
- Vector Spaces
- Vector Subspaces
- Linear Combinations, Span, spanning set

Let's warm up with two exercises:

Ex1: Is $H = \{ p(x) \in \mathbb{P}_n \mid p(1) = 0 \}$ – a subspace of $\mathbb{P}_n$?

Yes. You need to show two checks:

1) If $p(x), q(x) \in H$, then $p+q \in H$.
   Indeed: $(p+q)(1) = p(1) + q(1) = 0$
2) If $p(x) \in H$, $c \in \mathbb{R}$, then $c \cdot p(x) \in H$
   Indeed: $(c \cdot p)(1) = c \cdot p(1) = 0$ ∎

Ex2: Can the set of all linear transformations $T: \mathbb{R}^n \to \mathbb{R}^m$ be made into a vector space? What is the zero vector?

Step 1: Define addition and scalar multiplication

- If $T_1, T_2: \mathbb{R}^n \to \mathbb{R}^m$, set $T_1+T_2: \mathbb{R}^n \to \mathbb{R}^m$ via $\underset{\mathbb{R}^n}{\overset{\Downarrow}{\vec{v}}}$ $\vec{v} \longmapsto T_1(\vec{v}) + T_2(\vec{v})$
- If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}$, set $cT: \mathbb{R}^n \to \mathbb{R}^m$ via $\vec{v} \longmapsto c \cdot T(\vec{v})$

Step 2: Verify that thus defined T_1+T_2 and $c \cdot T$ are linear (given T_1, T_2, T - linear)

- $\underbrace{(T_1+T_2)(\vec{u}+\vec{v})}_{=} \overset{?}{=} (T_1+T_2)(\vec{u}) + (T_1+T_2)(\vec{v})$

$$T_1(\vec{u}+\vec{v}) + T_2(\vec{u}+\vec{v}) \underset{\substack{T_1,T_2 \\ \text{linear}}}{=} T_1(\vec{u}) + T_1(\vec{v}) + T_2(\vec{u}) + T_2(\vec{v}) = (T_1(\vec{u}) + T_2(\vec{u})) + (T_1(\vec{v}) + T_2(\vec{v}))$$

- $\underbrace{(T_1+T_2)(c\vec{v})}_{} \overset{?}{=} c(T_1+T_2)(\vec{v})$

$$T_1(c\vec{v}) + T_2(c\vec{v}) = c \cdot T_1(\vec{v}) + c \cdot T_2(\vec{v})$$

+ some verifications for $c \cdot T$

Step 3: Verify all the conditions on addition & scalar multiplication
   (skip in class)

The zero vector is the zero map $\mathbb{R}^n \to \mathbb{R}^m$ sending all $\vec{v} \mapsto \vec{0}$ ($\vec{v} \in \mathbb{R}^n$, $\vec{0} \in \mathbb{R}^m$) ∎

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- § 4.2 Null spaces, Column spaces, Row spaces, Linear Transformations

Def: The null space of an $m \times n$ matrix A, denoted Nul A, is the set of all solutions of $A\vec{x} = 0$, i.e.

$$\boxed{\text{Nul } A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}}$$

We already saw this concept back in §2.8!

The following is almost obvious:

CLAIM: Nul A is a subspace of $\mathbb{R}^n$ ← Q: Ask if it is clear to everyone.

Recall: algorithm for producing a spanning set of Nul A.

Def: The column space of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A, i.e.

$$\boxed{\text{Col } A = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} \quad \text{if } A = \left(\vec{a}_1 \ldots \vec{a}_n\right)}$$

Again, this concept was already discussed back in § 2.8.

Note: $\text{Col } A = \{\vec{b} \in \mathbb{R}^m \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$, hence,

Col A = range of the corresponding linear transformation

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longmapsto A\vec{x}.$$

Rem: $\dim(\text{Col } A) + \dim(\text{Nul } A) = \#\text{columns of } A$ as established in §2.9

The following is obvious:

CLAIM: Col A is a subspace of $\mathbb{R}^m$

Def: The row space of an $m \times n$ matrix A, denoted Row A, is the set of all linear combinations of the rows of A, i.e.

$$\boxed{\text{Row } A = \text{Span}\{\vec{b}_1, \ldots, \vec{b}_m\} \quad \text{if } A = \begin{pmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_m \end{pmatrix}}$$

CLAIM: Row A is a subspace of $\mathbb{R}^n$

Q: Any relation b/w Row A and Col A^T?
A: Coincide

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While Col A & Nul A provide the simplest constructions of subspaces of $\mathbb{R}^{?}$, it's natural to ask how they generalize once we replace $\mathbb{R}^n$ with more general vector spaces.

Def: A *linear transformation* T from a vector space V to a vector space W, denoted $T: V \to W$, is a rule that assigns to each $\vec{v} \in V$ a unique vector $T(\vec{v}) \in W$ so that

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in V$
2) $T(c\vec{u}) = c \cdot T(\vec{u})$ for any $\vec{u} \in V$ and scalar c

When $V=\mathbb{R}^n$, $W=\mathbb{R}^m$, this coincides with the old definition we had.

The *kernel* (a.k.a. *null space*) of T is $\{\vec{v} \in V \mid T(\vec{v}) = 0\}$

CLAIM: Kernel of T is a subspace of V

The *range* of T is $\{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\}$

CLAIM: Range of T is a subspace of W.

Example: $V=\mathbb{R}^n$, $W=\mathbb{R}^m$, $T=$ linear transf. with the standard matrix A.
Then: Kernel of $T =$ Nul A
Range of $T =$ Col A.

Ex 3: Consider $T: \mathrm{Mat}_{n\times n} \longrightarrow \mathrm{Mat}_{n\times n}$
$$A \longmapsto A + A^T$$
Verify that T is a linear transformation. Describe kernel & range of T.

Kernel of $T =$ skew-symmetric $n\times n$ matrices
Range of $T =$ symmetric $n\times n$ matrices

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• § 4.3 Linearly independent sets ; Bases

Similarly to the case of $\mathbb{R}^n$, we make the following definition

Def: An indexed set of vectors $\{\vec{v}_1, \ldots \vec{v}_k\}$ in a vector space V is said to be <u>linearly independent</u> if the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

has only the <u>trivial solution</u> $c_1 = c_2 = \ldots = c_k = 0$

Otherwise, the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be <u>linearly dependent</u>

Similarly to the discussion for $\mathbb{R}^n$, we have

> CLAIM: A set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ $(k \geq 2)$ with $\vec{v}_1 \neq \vec{0}$ is linearly dependent iff some $\vec{v}_j$ (with $1 < j \leq k$) is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

<u>Q</u>: When a set $\{\vec{v}_1\}$ is linearly dependent? ← only when $\vec{v}_1 = \vec{0}$.

<u>Ex 4</u>: a) Is the set $\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}2&3\\0&2\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ of $Mat_{2\times2}$ linearly independent?

b) Is the set $\{t^2-1, 2t+3, 5\}$ of $\mathbb{P}_2$ linearly independent?

c) Is the set $\{\cos t, \sin(t^2), 0, e^t\}$ of $C(\mathbb{R})$ linearly indep.?

a) No, as $(-2)\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix} + 1\cdot\begin{pmatrix}2&3\\0&2\end{pmatrix} + (-1)\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} = 0$

b) Yes: if $\underbrace{a(t^2-1) + b(2t+3) + c\cdot 5}_{at^2 - a + 2b\cdot t + 3b + 5c} = 0$, then $a = 0, b = 0, 5c - a + 3b = 0 \Rightarrow c = 0$

c) No: it contains 0, e.g. $1 \cdot 0 = 0$.

Def: Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a <u>basis for H</u> if

1) $\mathcal{B}$ is lin. indep. set

2) $H = \text{Span}\,\mathcal{B}$, i.e. subspace spanned by $\mathcal{B}$ coincides with H.

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Ex 5: Verify that

a) $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}_n$.

b) $\{1, x^2, x^4, \ldots, x^{2\lfloor \frac{n}{2} \rfloor}\}$ is a basis for the subspace of "even polynomials" in $\mathbb{P}_n$.

$\lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n}{2} & \text{if } n\text{-even} \\ \frac{n-1}{2} & \text{if } n\text{-odd} \end{cases}$

c) $\left\{ E_{ij} = m \left(\begin{array}{ccc} 0 & 1 & 0 \\ & 0 & \end{array} \right) \leftarrow i^{th} \text{ row} \right\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ – a basis for $Mat_{m \times n}$.

$n \quad \uparrow j^{th}$ column

Ex 6: 1) Find a basis of the subspace of symmetric matrices in $Mat_{n\times n}$

2) Find a basis of the subspace of skew-symm. matrices in $Mat_{n \times n}$

Ex 7: Evoking Ex 4a), find a basis of $Span\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}2&3\\0&2\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ in $Mat_{2\times 2}$.

As $\begin{pmatrix}0&1\\0&0\end{pmatrix} = -2\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix} + \begin{pmatrix}2&3\\0&2\end{pmatrix}$, any element in $Span\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}2&3\\0&2\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ also belongs to $Span\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}2&3\\0&2\end{pmatrix}\right\}$.

On the other hand, we claim that the set $\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}2&3\\0&2\end{pmatrix}\right\}$ is lin. indep, since if $a\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix} + b\begin{pmatrix}2&3\\0&2\end{pmatrix} = 0 \Rightarrow \begin{cases} a+2b=0 \\ a+3b=0 \end{cases} \Rightarrow a=b=0$.

So: $\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}2&3\\0&2\end{pmatrix}\right\}$ – a basis of the above Span ∎

Note: We could also use the same argument to prove that $\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}2&3\\0&2\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ – also bases.

Based on the same ideas, one proves the following result:

CLAIM: Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space V. Let H be the span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$, i.e. $H = Span\{\vec{v}_1, \ldots, \vec{v}_k\}$

a) If one of the vectors in S, say $\vec{v}_j$, is a linear combination of the remaining el-s in S, then the set formed from S by removing $\vec{v}_j$ still spans H

b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H