Lecture #15

- Last time → Linear transformations, their kernel and range
  → Linear dependent / independent sets of vectors
  → Bases for subspaces of a vector space

Ex1: a) Describe kernel and range of a linear transformation (verify it is linear)
$T: \mathbb{P}_3 \to \mathbb{R}^2$, defined via $T(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = (p_0 - p_1, p_2 - p_3)$

b) Describe kernel and range of a linear transformation
$T: Mat_{2\times 2} \to C(\mathbb{R})$, defined via $T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{12}\cdot x + a_{21}\cdot x^2 + a_{22} x^3$

Ex2: Prove that $\{\cos t, \sin t, e^t\}$ in $C(\mathbb{R})$ is linearly independent.

► Assume constants $a, b, c \in \mathbb{R}$ satisfy $a\cdot \cos t + b\cdot \sin t + c\cdot e^t = 0$ (zero function)
Plug $t = 0, 2\pi$ to get $a + c = 0$ and $a + e^{2\pi}\cdot c = 0$. The only solution of $\begin{cases} a + c = 0 \\ a + e^{2\pi}\cdot c = 0 \end{cases}$ is $a = c = 0$. Then $b\cdot \sin t = 0$ and plugging $t = \frac{\pi}{2}$, get $b = 0$

So: $a = b = c = 0$ ← the trivial solution ⇒ indeed lin. indep. ▪

Q: Does the above set $\{\cos t, \sin t, e^t\}$ form a basis of $C(\mathbb{R})$?

A: No, e.g. argue as above to show that $1 \notin \text{Span}\{\cos t, \sin t, e^t\}$

Rmk: While the definition of bases last time was given for subspaces of vector spaces, it's actually more natural to phrase what it means that you have a basis of a vector space (since a subspace of a vector space is naturally a vector space itself)

Bases of Col(A), Nul(A), Row(A)

- We discussed how to find bases of Col(A), Nul(A) before – RECALL
- As per Row(A), let's note that Row(A) does not change (unlike e.g. Col A) under elementary row operations, while for echelon form a basis is clear

⇒ $\boxed{\text{CLAIM: A set of nonzero rows in an echelon form of A is a basis for Row A}}$

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Ex3: Let $A = \begin{pmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 2 & 11 \end{pmatrix}$

a) Find a basis of Col A.

b) Find a basis of Row A

c) Find a basis of Nul A.

$$\begin{pmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 2 & 11 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 7 \\ 0 & 0 & -3 & -12 \\ 0 & 0 & 1 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \underline{1} & 2 & 0 & 3 \\ 0 & 0 & \underline{1} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Underlined are pivots (in the rightmost matrix).

So: the pivot columns are columns #1,3

a) So, $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ - basis of Col A.

To double-check, note they are clearly lin. ind., while $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 2 \\ 11 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

b) Even easier: $\{(1,2,0,3), (0,0,1,4)\}$ - basis of Row A.

c) Finally, let's find a basis of Nul A.

$\left.\begin{array}{l} x_4 \text{ - free} \Rightarrow x_3 = -4x_4 \\ x_2 \text{ - free} \Rightarrow x_1 = -2x_2 - 3x_4 \end{array}\right\} \Rightarrow$ solutions are $\begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix}$

Hence: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$ - basis of Nul A.

Rmk: Even though we skip Section 4.4, it's important to mention that once you have chosen a basis in a vector space, you can describe any vector in that vector space via coordinates.

! In particular, lin. dependence of $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ of $\mathrm{Mat}_{2\times 2}$ from last time is equivalent to lin. dependence of $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ of $\mathbb{R}^4$.

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- § 4.5 The dimension of a vector space

Claim: If a vector space V has a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, then any set in V containing more than n vectors must be linearly dependent

Claim: If a vector space V has a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, then any basis of V must contain exactly n vectors

Def: If a vector space V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V denoted $\dim V$, is the number of elements in any basis of V

Note: $\dim \{\vec{0}\} = 0$

Ex 4: Find dimensions of $\mathbb{P}_n$, $Mat_{m\times n}$, $Mat_{n\times n}^{symmetric}$, $Mat_{n\times n}^{skew\text{-}symmetric}$.

Ex 5: What are the dimensions of Col A, Row A, Nul A from Ex 3.

Q: How those can be described via pivots?

Recall: rank $A := \dim \text{Col } A = \#$ pivot columns $\stackrel{!}{=} \dim \text{Row } A$

& nullity $A := \dim \text{Nul } A = \#$ non-pivot columns $\Big\} \Rightarrow$

$\Rightarrow$ **Claim** ("Rank Theorem"): rank A + nullity A = # columns in A.

Ex 6: Can a 5×9 matrix have nullity 3?

If nullity $= 3 \Rightarrow$ rank $= 9-3=6$.

But Col A is a subspace of $\mathbb{R}^5$, hence, $\dim \text{Col } A \leq 5$ (see claim below)

Contradiction! SO: the answer is "NO"

Claim: Let H be a subspace of a finite-dimensional vector space V.

1) Any lin. indep. set in H can be expanded to a basis for H
2) $\dim H \leq \dim V$

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Another useful result is generalizing its counterpart for $\mathbb{R}^n$:

Claim ("Basis Theorem"): Let V be an n-dimensional vector space.
1) Any lin. indep. set of n elements in V is automatically a basis of V
2) Any set of n elements in V that spans V is automatically a basis of V

Ex 7 (see Example 8 on p.246 of the textbook): Given a nonhomogeneous system of 40 linear equations in 42 variables, such that the corresponding homogeneous system has 2-dim solution set, prove the original non-homogeneous system is consistent.

► nullity $= 2 \Rightarrow$ rank $= 40 = \#$ columns $\Rightarrow \operatorname{Col} A = \mathbb{R}^{40} \Rightarrow A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$

Combining the above considerations, let's conclude with the following result:

Claim: Let $A \in \mathrm{Mat}_{n\times n}$. The following are equivalent to "A-invertible":
1) $\operatorname{Col} A = \mathbb{R}^n$
2) $\operatorname{rank} A = n$
3) $\operatorname{Nul} A = \{0\}$
4) $\operatorname{nullity} A = 0$