Lecture #16

• Last time → Bases for Col A, Row A, Nul A

→ Dimension of a vector space

→ "Basis theorem"

Remark: Since given a spanning set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V, a certain subset of it forms a basis, we have

$$\boxed{\dim V \leq n}$$

Remark: Given a finite-dimensional vector space V, and a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$, note that every $\vec{v} \in V$ can be uniquely written as

$$\boxed{\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n}$$

The corresponding map

$$\boxed{\begin{array}{ccc} V & \longrightarrow & \mathbb{R}^n \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \vec{v} & \longmapsto & \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \end{array}}$$

is 1-to-1, and thus we may identify V with $\mathbb{R}^n$, so that addition and scalar multiplication on V correspond to those on $\mathbb{R}^n$.

In particular, linear dependence of

$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\} \in Mat_{2\times 2}$$

is equivalent to linear dependence of

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\} \in \mathbb{R}^4.$$

Ex1: a) What is the maximal rank of a 4×6 matrix?

b) — // — 6×4 matrix?

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• §5.1 Eigenvectors & eigenvalues

Ok, now back to practical computations, for today & next time we will focus on eigenvectors/ eigenvalues and their applications.

Def: An *eigenvector* of a square $n\times n$ matrix A is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ, called an *eigenvalue* of A.

Note: λ may be 0, BUT $\vec{v}$ is supposed to be nonzero.

Example: A vector $\begin{pmatrix}1\\-1\end{pmatrix}$ is an eigenvector of $A=\begin{pmatrix}5&3\\-1&1\end{pmatrix}$ with eigenvalue 2 as $\begin{pmatrix}5&3\\-1&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\end{pmatrix}=2\cdot\begin{pmatrix}1\\-1\end{pmatrix}$

Q: But how can we find the eigenvectors & eigenvalues for a given matrix A?

Ex 2: Prove that $\lambda=4$ is also an eigenvalue of the same $A=\begin{pmatrix}5&3\\-1&1\end{pmatrix}$ Find the corresponding eigenvector.

4 is an eigenvalue iff $A\vec{v}=4\vec{v}$ has a nontrivial solution.

Equivalently, the homogeneous eq-n $\underbrace{(A-4I_2)}_{=\begin{pmatrix}1&3\\-1&-3\end{pmatrix}}\vec{v}=0$ has a nontriv. sol-n.

The columns of $\begin{pmatrix}1&3\\-1&-3\end{pmatrix}$ are clearly lin. dep., and the general solution has the form $x_3\cdot\begin{pmatrix}-3\\1\end{pmatrix}$

So: We see that any nonzero multiple of $\begin{pmatrix}-3\\1\end{pmatrix}$ is an eigenvector with eigenvalue 4. ∎

As follows from this proof, we have:

$$\boxed{\text{Claim: } \lambda - \text{eigenvalue of } A\in \mathrm{Mat}_{n\times n} \text{ iff } (A-\lambda I_n)\vec{v}=0 \text{ has a nontrivial solution.}}$$

We already discussed many different criterias to check that

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The set of all eigenvectors with eigenvalue λ is the same as the null-space of $A-\lambda I_n$.

|| Def: This set is called the eigenspace of A corresponding to λ

Ex3: Find all eigenvalues and corresponding eigenspaces of $A=\begin{pmatrix}5&3\\-1&1\end{pmatrix}$

▶ $\det\begin{pmatrix}5-\lambda & 3\\ -1 & 1-\lambda\end{pmatrix}=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4) \Rightarrow$ zero at $\lambda=2$, $\lambda=4$.

We saw that eigenspace of A corresponding to $\lambda=4$ is the line through $(-3,1)$

Likewise, the eigenspace of A corr. to $\lambda=2$ is the line through $(1,-1)$

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Remark: If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$, which consists of eigenvectors of $A\in Mat_{n\times n}$ (i.e. $A\vec{v}_i=\lambda_i\vec{v}_i$), then you can think of A as dilation by λ_i in the direction of $\vec{v}_i$.

Ex4: Find an eigenspace of $A=\begin{pmatrix}4&2&3\\1&5&3\\1&2&6\end{pmatrix}$ corresponding to $\lambda=3$ and its basis.

▶ $A-3I_3=\begin{pmatrix}1&2&3\\1&2&3\\1&2&3\end{pmatrix} \xrightarrow[\text{reduct.}]{\text{row}} \begin{pmatrix}1&2&3\\0&0&0\\0&0&0\end{pmatrix}$

$$\left.\begin{array}{l} x_2, x_3 - \text{free} \\ x_1=-2x_2-3x_3\end{array}\right\} \Rightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-2x_2-3x_3\\x_2\\x_3\end{pmatrix}=x_2\begin{pmatrix}-2\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

So: $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$ – basis of that eigenspace

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Rmk: It's always a good idea to verify that eigenvectors you found are correct, just by direct evaluation of $A\vec{v}$.

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Claim: The eigenvalues of a triangular matrix are the entries on its main diagonal

↑ Discuss why (if time permits)

Ex 5: Find eigenvalues and eigenspaces of
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Claim: If $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors of A with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ – lin. indep. set

As we shall see soon, the ideal situation is when one can find a basis consisting of eigenvectors, BUT this is not always possible!

Solution of Ex 5

a) $A - \lambda I_3 = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} \Rightarrow \det(A - \lambda I_3) = (1-\lambda)^3 \Rightarrow$ the only eigenvalue is $\lambda = 1$.

$\lambda = 1 \Rightarrow$ get $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \Rightarrow x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ – the null-space!

b) $\det(A - \lambda I_3) = (1-\lambda)(2-\lambda)(3-\lambda) \Rightarrow$ eigenvalues: $\lambda = 1, 2, 3$

$\boxed{\lambda=1}$ $A - 1 \cdot I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ solutions are: $x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\boxed{\lambda=2}$ $A - 2I_3 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ solutions are: $x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\boxed{\lambda=3}$ $A - 3I_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ solutions are: $x_3 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$ ■