Lecture #17

- Last time → Eigenvectors, eigenvalues, eigenspaces

Ex1: Find eigenvalues and eigenspaces of

a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

▶ a) A – triangular, on diagonal have 1 everywhere ⇒ the only eigenvalue is $\lambda = 1$.

$A - 1 \cdot Id = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow (A - 1 \cdot Id)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 0 \end{pmatrix}$ – vanishes iff $x_2 = x_3 = 0$.

Hence: eigenspace corresponding to $\lambda = 1$ is a line $x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

b) A – triangular, on diagonal have 1, 2, 3.

$\lambda = 1 \Rightarrow A - \lambda Id = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_2 + x_3 \\ 2x_3 \end{pmatrix}$ – vanishes iff $x_2 = x_3 = 0$

Hence: eigenspace corresp. to $\lambda = 1$ is a line $x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2 \Rightarrow A - \lambda Id = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} -x_1 + x_2 \\ x_3 \\ x_3 \end{pmatrix}$ – vanishes iff $x_1 = x_2$ & $x_3 = 0$

Hence: eigenspace corresponding to $\lambda = 2$ is a line $x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3 \Rightarrow A - \lambda Id = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} -2x_1 + x_2 \\ -x_2 + x_3 \\ 0 \end{pmatrix}$ – vanishes iff $x_3 = x_2 = 2x_1$

Hence: eigenspace corresponding to $\lambda = 3$ is a line $x_3 \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

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Warning: As we see in part a), there is no basis consisting of A-eigenvectors.

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• § 5.2 The Characteristic Equation

As we saw last time: λ - eigenvalue of $A \in Mat_{n\times n} \Leftrightarrow A - \lambda \cdot I_n$ - not invertible.

But: $A - \lambda I_n$ - not invertible $\Leftrightarrow \det(A - \lambda I_n) = 0$.

This brings us to the equation in λ:

$$\underbrace{\boxed{\det(A - \lambda \cdot I_n) = 0}}_{\text{degree } n \text{ polynomial in } \lambda} \leftarrow \text{called the characteristic equation}$$

Claim: A scalar λ is an eigenvalue of an $n\times n$ matrix iff λ satisfies
$$\det(A - \lambda I_n) = 0$$

Ex 2: Find the characteristic equations in Ex1.

a) $(1-\lambda)^3 = 0$

b) $(1-\lambda)(2-\lambda)(3-\lambda) = 0$

As observed above, $\det(A - \lambda I_n)$ is a degree n polynomial in λ, called the characteristic polynomial of A.

So λ - eigenvalue of A iff λ is a root of its charact. polynomial

Def: The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example: In Ex1(a), $\lambda = 1$ has alg. multiplicity $= 3$

Ex1(b), $\lambda = 1, 2, 3$ have alg. multiplicity $= 1$

Ex3: The characteristic polynomial of a 5×5 matrix A equals $-\lambda^5 + 2\lambda^3 - \lambda$. Find the eigenvalues and their multiplicities.

$$-\lambda^5 + 2\lambda^3 - \lambda = -\lambda(\lambda^2 - 1)^2 = -\lambda(\lambda - 1)^2(\lambda + 1)^2.$$

So: Eigenvalues are $0, 1, -1$ with multiplicities $1, 2, 2$, resp.

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It is known that any degree n polynomial has exactly n complex roots counting multiplicities. While we shall discuss complex numbers in a while, until then we can only use above result as: # real roots $\leq n$.

Ex 4: Find real eigenvalues of $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ for $0 < \varphi < \pi$.

$$\det(A - \lambda I_2) = \begin{vmatrix} \cos\varphi - \lambda & -\sin\varphi \\ \sin\varphi & \cos\varphi - \lambda \end{vmatrix} = \lambda^2 - 2\cos\varphi \cdot \lambda + \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1}$$

which has no real roots as it may be written as $\underbrace{(\sin\varphi)^2}_{>0} + \underbrace{(\cos\varphi - \lambda)^2}_{\geq 0}$

$\Rightarrow$ NO real eigenvalues $\blacksquare$

Q: Can you explain the absence of real eigenvalues geometrically?

Def: If A, B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, equivalently, $A = PBP^{-1}$

Note: If A is similar to B, then B is similar to A.

! Hence, we just say that "A and B are similar".

Def: Changing A into $P^{-1}AP$ is called a similarity transformation

As $P^{-1}AP - \lambda I_n = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I_n)P$ and $\det(XY) = \det X \cdot \det Y$, we get:

Claim: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial, and hence the same eigenvalues (with the same multiplicities)

Ex 5: a) Verify that $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ have the same charact. polynomials.

b) Show that no two of them are similar.

Hint: Compute dim of $\lambda = 1$ eigenspace.

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- § 5.3 Diagonalization

Ex 6: Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Find A^3

Clearly: $A^3 = \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}$ ▨

Ex 7: Let $B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Find B^k for any k.

Hint: $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{P}\underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}}_{P^{-1}}$

So: $B^k = (PA\cancel{P^{-1}})(\cancel{P}A\cancel{P^{-1}})(\cancel{P}A\cancel{P^{-1}}) \cdots (PAP^{-1}) = \ldots = P \cdot A^k \cdot P^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^k & 3^k \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{pmatrix}}_{\text{ANSWER}}$$ ▨

Def: A square $n \times n$ matrix is said to be *diagonalizable* if A is *similar* to a diagonal matrix D, i.e.

$A = PDP^{-1}$, P - invertible matrix, D - diagonal matrix

Claim: An $n \times n$ matrix A is diagonalizable iff A has n linearly indep. eigenvectors

Moreover, we have

Claim: $A = PDP^{-1}$ with D - diagonal iff the columns of P are n lin. indep. eigenvectors of A, in which case the diagonal entries of D are eigenvalues of A that correspond to these eigenvectors

Def: Given $A \in Mat_{n \times n}$, a basis of $\mathbb{R}^n$ consisting of eigenvectors of A is called an *eigenvector basis*

Ex 8: Recover the decomposition $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$ used in the proof of Ex 7.