Lecture # 18

- Last time → Characteristic equation, characteristic polynomial
  - → multiplicity of eigenvalues
  - → Similarity transformation $A \rightsquigarrow P^{-1}AP$
  - → Diagonalizable square matrices ($A = PDP^{-1}$, D-diagonal)

Ex 1: Is $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ diagonalizable?

▸ Char. polynomial of A is $(1-\lambda)^4 \Rightarrow$ only eigenvalue $\lambda = 1$ (multiplicity $= 4$)
If A was diagonalizable, i.e. $A = PDP^{-1}$ with D-diagonal, then char. pol-l of D should be the same, i.e. $(1-\lambda)^4$. But as D-diagonal, this implies all diagonal entries of D are 1.
Hence, $D = I_4 \Rightarrow PDP^{-1} = P \cdot I_4 \cdot P^{-1} = I_4 \neq A \Rightarrow$ Contradiction.

So: A is not diagonalizable.

Last time, we also had the following two claims at the very end:

Claim: An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors

Claim: We have $A = PDP^{-1}$ with D-diagonal iff the columns of P are n linearly indep. eigenvectors of A, while the diagonal entries of D are the corresponding eigenvalues

Def: Given an $n \times n$ matrix A, a basis of $\mathbb{R}^n$ consisting of eigenvectors of A is called an *eigenvector basis of $\mathbb{R}^n$*.

Combining 1st claim with lin. indep. of eigenvectors for different eigenvalues, get:

Claim: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Ex2: Diagonalize the matrix $A=\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, if possible.

<u>Step 1</u>: Find eigenvalues.

A - triangular $\Rightarrow$ eigenvalues are $\lambda = 2, 3$.

<u>Step 2</u>: Find lin. ind. eigenvectors

$\underline{\lambda=2}$: $A - 2I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix} \Rightarrow$ eigenvectors are $\begin{pmatrix} a \\ 0 \end{pmatrix}$

Can take $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\underline{\lambda=3}$: $A - 3I_2 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x+y \\ 0 \end{pmatrix} \Rightarrow$ eigenvectors are $\begin{pmatrix} a \\ a \end{pmatrix}$.

Can take $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

<u>Step 3</u>: Construct P from above eigenvectors.

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

<u>Step 4</u>: Construct D from eigenvalues

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

<u>So</u>: $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$ $\leftarrow$ This was used in <u>Ex7</u> in Lecture #17

Q: Can someone suggest an alternative proof of Ex 1?

$\lambda = 1$ - the only eigenvalue

$$A - \lambda I_4 = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} \Rightarrow \text{eigenspace} = \left\{ \begin{pmatrix} * \\ 0 \\ * \\ 0 \end{pmatrix} \right\} \text{ - 2-dim}$$

$\Rightarrow$ can not pick an eigenbasis!

<u>Remarks</u>: (1) We can change the order of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so that $P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

<u>BUT</u> then D is also changed: $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, i.e. $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$

(2) We could also pick other bases, e.g. $P = \begin{pmatrix} 99 & 7 \\ 0 & 7 \end{pmatrix}$, so that $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 99 & 7 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 99 & 7 \\ 0 & 7 \end{pmatrix}^{-1}$.

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Ex 3: Diagonalize the matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$, if possible.

▶ Step 1: Find eigenvalues.

$\lambda = 1, 2, 3, 4.$

Step 2: Find eigenvectors.

$\underline{\lambda = 1} \Rightarrow A - \lambda I_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_2 \\ 2x_3 + x_4 \\ 3x_4 \end{pmatrix}$ – vanishes $\underline{\underline{\text{iff}}}$ $x_2 = x_3 = x_4 = 0$

Can take $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\lambda = 2} \Rightarrow A - \lambda I_4 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} -x_1 + x_2 \\ 0 \\ x_3 + x_4 \\ 2x_4 \end{pmatrix}$ – vanishes $\underline{\underline{\text{iff}}}$ $x_1 = x_2$ & $x_3 = x_4 = 0$

Can take $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\lambda = 3} \Rightarrow A - \lambda I_4 = \begin{pmatrix} -2 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{O}} \\ 0 & 1 & & \\ \multicolumn{2}{c}{\multirow{2}{*}{O}} & 0 & 1 \\ & & 0 & 1 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} -2x_1 + x_2 \\ x_2 \\ x_4 \\ x_4 \end{pmatrix}$ – vanishes $\underline{\underline{\text{iff}}}$ $x_1 = x_2 = x_4 = 0$

Can take $\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{\lambda = 4} \Rightarrow A - \lambda I_4 = \begin{pmatrix} -3 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{O}} \\ 0 & -2 & & \\ \multicolumn{2}{c}{\multirow{2}{*}{O}} & -1 & 1 \\ & & 0 & 0 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} -3x_1 + x_2 \\ -2x_2 \\ -x_3 + x_4 \\ 0 \end{pmatrix}$ – vanishes $\underline{\underline{\text{iff}}}$ $x_1 = x_2 = 0$ & $x_3 = x_4$

Can take $\vec{v_4} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Step 3: Construct P

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 4: Construct D

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

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Let us conclude this discussion with the following criteria when a square matrix with less than n distinct eigenvalues is diagonalizable:

Claim: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_k$.

1) The dimension of the eigenspace for λ_j $(1 \le j \le k)$ is at most the multiplicity of the eigenvalue λ_j.
2) A is diagonalizable iff
   - the characteristic polynomial factors into linear factors

     AND
   - the dimension of the eigenspace for each λ_j equals exactly the multiplicity of λ_j.
3) If 2) holds, and $\mathcal{B}_j$ is a basis for the eigenspace corresponding to λ_j for each $1 \le j \le k$, then the union of all $\mathcal{B}_j$ $(1 \le j \le k)$ forms an eigenvector basis for $\mathbb{R}^n$.

Remarks: (1) In Exercise 1, we have $\lambda = 1$ with multiplicity 4, while the dimension of the eigenspace for $\lambda = 1$ is 2. As $2 < 4$, get A - non diagonalizable

(2) In Exercise 3, have 4 distinct eigenvalues $\Rightarrow$ know right away that A is diagonalizable

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T/F: A degenerate (singular) 4×4 matrix has an eigenvalue $\lambda = 0$

True: $\lambda = 0$ - eigenvalue $\Leftrightarrow \det(A - 0 \cdot I_4) = 0 \Leftrightarrow \det A = 0 \Leftrightarrow A$ - singular ▣

T/F: A 4×4 matrix can have 5 distinct eigenvalues.

False: Characteristic polynomial is of degree 4 $\Rightarrow$ has at most 4 roots $\Rightarrow$ there are at most 4 eigenvalues ▣

T/F: If A is similar to B, then A^3 is similar to B^3

True: If $A = PBP^{-1} \Rightarrow A^3 = PB\cancel{P^{-1}} \cdot \cancel{P}B\cancel{P^{-1}} \cdot \cancel{P}BP^{-1} = P \cdot B^3 \cdot P^{-1}$ ▣

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• §5.4 Eigenvectors and linear transformations

Let's generalize the above discussion, by replacing:

- $\mathbb{R}^n$ with a general vector space V
- $n \times n$ matrix A with a linear transformation $T: V \longrightarrow V$.

Def: Let V be a vector space. An eigenvector of a linear transf. $T: V \longrightarrow V$ is a nonzero $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \cdot \vec{v}$ for some scalar λ, called an eigenvalue of T

Examples: 1) $V = \underbrace{C^1(\mathbb{R})}_{\text{differentiable } f\text{-s}}$, $T: f(x) \longmapsto f'(x)$ - linear operator.

Then: $f(x) = e^{\lambda x}$ is an eigenvector with eigenvalue λ.

2) $V = \mathbb{S}$, $T: (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \longmapsto (\ldots, y_{-1}, y_0, y_1, y_2, \ldots)$

Then: $(\ldots, \frac{1}{\lambda^2}, \frac{1}{\lambda}, 1, \lambda, \lambda^2, \ldots) \in \mathbb{S}$ is an eigenvector with eigenvalue λ.

Ex 4: Find all eigenvalues of $T: \mathbb{P}_n \rightarrow \mathbb{P}_n$ defined via $p(x) \longmapsto p'(x)$.

▶ If $p(x)$ is a degree $k \geq 1$ polynomial, then $p'(x)$ is a degree $k-1$ pol-l. Hence, it may not happen that $p'(x) = \lambda \cdot p(x)$ unless $p(x)$ is a degree 0 polynomial, i.e. $p(x) = p_0$ - constant. Then $p'(x) = 0 = 0 \cdot p(x)$.

So: The only eigenvalue is $\underline{\lambda = 0}$. ■

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For the rest of today, we assume V to be fin. dimensional!

Choose a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V. Then, for any $\vec{x} \in V$, recall the coordinate vector $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^n$ defined as $\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$ with $\vec{x} = r_1\vec{b}_1 + \ldots + r_n\vec{b}_n$.

We also have $[T(\vec{x})]_{\mathcal{B}} \in \mathbb{R}^n$.

$\boxed{Q}$: What is the exact relation b/w $[\vec{x}]_{\mathcal{B}}$ and $[T(\vec{x})]_{\mathcal{B}}$?

If $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \Rightarrow \vec{x} = r_1\vec{b}_1 + \ldots + r_n\vec{b}_n \Rightarrow T(\vec{x}) = r_1 T(\vec{b}_1) + \ldots + r_n T(\vec{b}_n)$

$$\Rightarrow [T(\vec{x})]_{\mathcal{B}} = r_1 [T(\vec{b}_1)]_{\mathcal{B}} + \ldots + r_n [T(\vec{b}_n)]_{\mathcal{B}} = M \cdot [\vec{x}]_{\mathcal{B}},$$

where M is an $n \times n$ matrix given by $M = \begin{pmatrix} [T(\vec{b}_1)]_{\mathcal{B}} & \ldots & [T(\vec{b}_n)]_{\mathcal{B}} \end{pmatrix}$

‖ Def: matrix for T relative to the basis $\mathcal{B}$
also denoted $[T]_{\mathcal{B}}$.

Ex5: In the setup of Ex4, determine $[T]_{\mathcal{B}}$, where $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$.

$$[T]_{\mathcal{B}} = \underbrace{\begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \ddots \\ & & & 0 & n \\ & & & & 0 \end{pmatrix}}_{n+1} \Big\updownarrow n+1$$

In the particular case $V = \mathbb{R}^n$, we have a natural identification b/w $\{$linear transformations $\mathbb{R}^n \to \mathbb{R}^n\}$ and $\{n \times n$ matrices$\}$.

In particular, our previous discussion yields:

Claim: Suppose $A = PDP^{-1}$, P - invertible $n \times n$ matrix, D - diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed by the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\vec{x} \mapsto A\vec{x}$.

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More generally, if $A \& C$ are similar, i.e.

$$A = PCP^{-1}$$

then C is the $\mathcal{B}$-matrix for the transformation $\vec{x} \mapsto A\vec{x}$ when the basis $\mathcal{B}$ is formed by the columns of P.

! The converse is also true.

So, we get:

Claim: The set of all matrices similar to $A \in Mat_{n \times n}$ coincides with the set of all matrix representations of the linear transformation $\vec{x} \mapsto A\vec{x}$.

Important Technical Remark !!!

To compute $P^{-1}AP$, one does not need actually to evaluate P^{-1}!

Instead:

1) Compute $A \cdot P$
2) Row reduce $[P \,\vdots\, AP]$ to $[I_n \,\vdots\, P^{-1}AP]$