Lecture #19

Last time $\longrightarrow$ Eigenvectors / eigenvalues of linear transformations $T: V \to V$

$\longrightarrow$ The notion of a $\underbrace{\text{matrix for } T \text{ relative to the basis } \mathcal{B}}_{[T]_{\mathcal{B}}}$.

Recall: If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis of V, then any $\vec{x} \in V$ can be uniquely written as $\vec{x} = r_1\vec{b}_1 + r_2\vec{b}_2 + \dots + r_n\vec{b}_n$, which allows to identify

$$V \xrightarrow{\sim} \mathbb{R}^n$$

$$\vec{x} \longmapsto \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} =: \underbrace{[\vec{x}]_{\mathcal{B}}}_{\text{coordinate vector}}$$

Key:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\wr"] & V \arrow[d, "\wr"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

linear transformation determined by the matrix

$$\boxed{[T]_{\mathcal{B}} = \left([T(\vec{b}_1)]_{\mathcal{B}} \quad [T(\vec{b}_2)]_{\mathcal{B}} \quad \dots \quad [T(\vec{b}_n)]_{\mathcal{B}} \right)}$$

Ex 1: Let V be a 2-dimensional vector space with a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$. Let $T: V \to V$ be a linear transformation with $T(\vec{b}_1) = 3\vec{b}_1 + 2\vec{b}_2$, $T(\vec{b}_2) = -5\vec{b}_1 - 6\vec{b}_2$.

(a) Find $[T]_{\mathcal{B}}$

(b) Find $T(10\vec{b}_1 + 3\vec{b}_2)$

▸ (a) Obvious: $[T]_{\mathcal{B}} = \begin{pmatrix} 3 & -5 \\ 2 & -6 \end{pmatrix}$

(b) 1st Proof: $T(10\vec{b}_1 + 3\vec{b}_2) \overset{\text{linear}}{=} 10\,T(\vec{b}_1) + 3\,T(\vec{b}_2) = 10(3\vec{b}_1 + 2\vec{b}_2) + 3(-5\vec{b}_1 - 6\vec{b}_2) = 15\vec{b}_1 + 2\vec{b}_2$

2nd Proof: $\begin{pmatrix} 3 & -5 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \Rightarrow T(10\vec{b}_1 + 3\vec{b}_2) = 5\vec{b}_1 + 2\vec{b}_2$ ■

Ex 2: Consider $T: \mathbb{P}_2 \to \mathbb{P}_2$ via $T(a_0 + a_1 t + a_2 t^2) = (3a_0 + 4a_1) + (a_0 - a_1 + a_2)t + (2a_0 + a_2)t^2$.
Find $[T]_{\mathcal{B} = \{1, t, t^2\}}$.

▸ $T(1) = 3 + t + 2t^2$, $T(t) = 4 - t$, $T(t^2) = t + t^2$ $\Rightarrow [T]_{\{1,t,t^2\}} = \begin{pmatrix} 3 & 4 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ ■

! Make sure you know the material from the end of Lecture 18!

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• § Appendix B: "Complex Numbers"

Def: A complex number is a number written in the form
$$z = a + b \cdot i$$
where $a, b \in \mathbb{R}$ and i is a formal symbol satisfying $i^2 = -1$.

a – real part of z, denoted $\operatorname{Re} z$

b – imaginary part of z, denoted $\operatorname{Im} z$

So: $z = 0$ iff $a = b = 0$, equivalently, $z_1 = z_2$ iff $\operatorname{Re} z_1 = \operatorname{Re} z_2$ & $\operatorname{Im} z_1 = \operatorname{Im} z_2$

Def: $\mathbb{C}$ – the set of all complex numbers

Rem: Any real number $a \in \mathbb{R}$ is considered as a special case via $a + 0 \cdot i$.

Key: $\mathbb{C}$ is endowed with addition & multiplication:

$$\boxed{\begin{aligned} (a+bi) + (c+di) &= (a+c) + (b+d)i \\ (a+bi)(c+di) &= (ac - bd) + (ad + bc)i \end{aligned}}$$

Subtraction is also clear:

$$\boxed{(a+bi) - (c+di) = (a-c) + (b-d)i}$$

Def: The conjugate of $z = a + bi$ is $\bar{z} := a - bi$.

Note: If $z = a + bi$, then $z \cdot \bar{z} = (a+bi)(a-bi) = a^2 + b^2 \in \mathbb{R}_{\geq 0}$.

Def: The absolute value (a.k.a. modulus) of $z = a + bi$ is $|z| := \sqrt{a^2 + b^2}$.

Properties:
1) $z = \bar{z}$ iff $z \in \mathbb{R}$ (i.e. $\operatorname{Im} z = 0$)
2) $\overline{z+w} = \bar{z} + \bar{w}$
3) $\overline{zw} = \bar{z} \cdot \bar{w}$
4) $z \cdot \bar{z} = |z|^2 \geq 0$
5) $|zw| = |z| \cdot |w|$
6) $|z + w| \leq |z| + |w|$

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! If $z \in \mathbb{C}$ and $z \neq 0$, then z has a multiplicative inverse $1/z = z^{-1} := \frac{\bar{z}}{|z|^2}$,

i.e. $z \cdot z^{-1} = z^{-1} \cdot z = 1$.

This allows to divide $\underbrace{z/w}_{:= z \cdot \frac{1}{w}}$ for two complex numbers $z, w \in \mathbb{C}$ with $w \neq 0$

Ex 3: Let $z = 1+2i$, $w = 3+4i$.

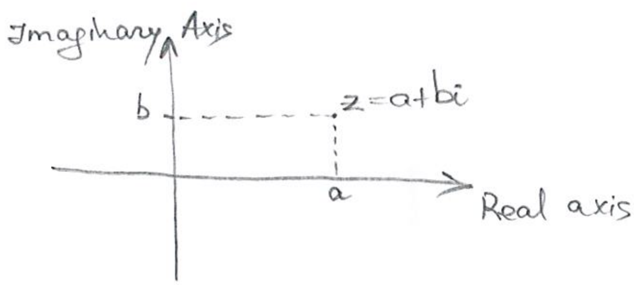
a) Compute $|z|$, $|w|$.

b) Compute $z+w$, $z \cdot w$.

c) Compute $\frac{z}{w}$.

Geometric Interpretation

Each $z = a+bi \in \mathbb{C}$ can be naturally depicted by the point (a,b)



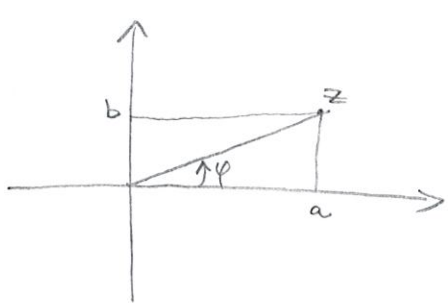
Q: What is the geometric meaning of $|z|$?

Q: What is the geometric meaning of the conjugate $\bar{z}$?

Q: What is the geometric meaning of the addition of two complex numbers?

To give a geometric representation of complex multiplication, one needs to use polar coordinates in $\mathbb{R}^2$.

Def: The argument of $z = a+bi$ is the angle φ between the positive real axis and the point (a,b) with conventions $-\pi < \varphi \leq \pi$.



Note: $\boxed{\begin{array}{l} a = |z| \cos\varphi \\ b = |z| \sin\varphi \end{array}} \Rightarrow \boxed{z = |z| (\cos\varphi + i \sin\varphi)}$

! Notation: We shall use the notation $e^{i\varphi}$ to denote $\cos\varphi + i \cdot \sin\varphi$

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Ex 4: Let $z = |z|(\cos\varphi + i\sin\varphi)$, $w = |w|(\cos\theta + i\sin\theta)$. Compute $z\cdot w$.

$$z\cdot w = |z|\cdot|w|\Big(\underbrace{(\cos\varphi\cdot\cos\theta - \sin\varphi\cdot\sin\theta)}_{\cos(\varphi+\theta)} + i\underbrace{(\cos\varphi\cdot\sin\theta + \sin\varphi\cdot\cos\theta)}_{\sin(\varphi+\theta)}\Big)$$

$$= |z|\cdot|w|\big(\cos(\varphi+\theta) + i\sin(\varphi+\theta)\big)$$

□

<u>So</u>: The product of two nonzero complex numbers is given in polar form by the product of their absolute values and the sum of their arguments.

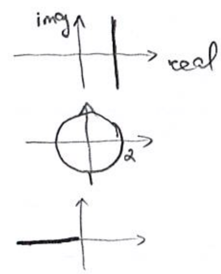
<u>Similarly</u>: If $w \neq 0$, then $\boxed{\frac{z}{w} = \frac{|z|}{|w|}\big(\cos(\varphi-\theta) + i\sin(\varphi-\theta)\big)}$

<u>Rmk</u>: Multiplication by $e^{i\varphi} = \cos\varphi + i\sin\varphi$ is just rotation by $\circlearrowleft\varphi$.

Ex 5: a) Depict all complex numbers z with $\operatorname{Re} z = 1$

b) —"— with $|z| = 2$

c) —"— with argument $= \pi$



An important application of the above product formula is:

if $z = r(\cos\varphi + i\sin\varphi)$, then $\boxed{z^k = r^k(\cos(k\varphi) + i\sin(k\varphi))}$ ← <u>De Moivre's Theorem</u>

Ex 6: Find all complex solutions of $z^5 = 2^5$.

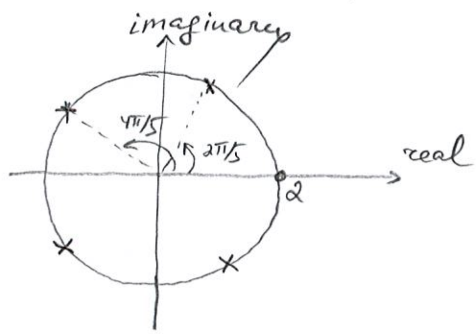
Let $z = r(\cos\varphi + i\sin\varphi) \Rightarrow z^5 = r^5(\cos(5\varphi) + i\sin(5\varphi))$

<u>So</u>: $z^5 = 2^5$ <u>iff</u> $r = 2$ and $5\varphi = 2\pi k$ with k – integer

As $-\pi < \varphi \leq \pi$, the solutions of $5\varphi = 2\pi k$ are: $\varphi = -\frac{4\pi}{5}, -\frac{2\pi}{5}, 0, \frac{2\pi}{5}, \frac{4\pi}{5}$.

<u>Thus</u>: $z = 2\left(\cos\left(\frac{2\pi k}{5}\right) + i\sin\left(\frac{2\pi k}{5}\right)\right)$, $\quad -2 \leq k \leq 2$ ← integer

<u>Graphically</u>:



! <u>Note</u>: Changing k by multiples of 5 doesn't change z.

□

<u>Rmk</u>: The product on $\mathbb{C}$ is not something we have on the side of $\mathbb{R}^2$