Lecture #20

Last time: $\underbrace{\mathbb{C}}_{\text{set of complex numbers}} = \{ z = \underbrace{a}_{\text{real}} + \underbrace{b \cdot i}_{\text{purely imaginary}} \mid a, b \in \mathbb{R} \}$

- Note: $\mathbb{R}$ is a subset of $\mathbb{C}$ via $\underset{\mathbb{R}}{a} \mapsto \underset{\mathbb{C}}{a + 0 \cdot i}$
- Know: how to compute: $z+w$, $z-w$, $z \cdot w$, z/w for $z, w \in \mathbb{C}$
- Complex conjugate: $z = a + ib \rightsquigarrow \bar{z} = a - ib$

  properties: $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$, $z \cdot \bar{z} = |z|^2$
- Know: geometric meaning of $z \cdot w$, $\frac{z}{w}$ (using polar coordinates)
- Def: For $\varphi \in \mathbb{R}$, use $e^{i\varphi}$ to denote $e^{i\varphi} = \cos(\varphi) + i \sin\varphi$

Then: If $z = r(\cos\varphi + i \sin\varphi) = re^{i\varphi}$

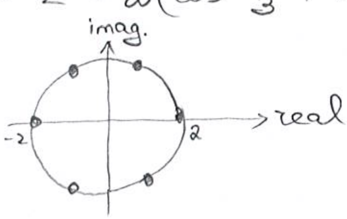
$\Downarrow$

$$\boxed{z^k = r^k(\cos k\varphi + i \sin k\varphi) = r^k e^{i \cdot k\varphi}}$$ ← De Moivre's Theorem

Ex1: Find all real and complex solutions of $z^6 = 2^6$

Real: $z = \pm 2$

Cpx: $z = 2\left(\cos \frac{\pi k}{3} + i \sin \frac{\pi k}{3}\right)$ with $k = -2, -1, 0, 1, 2, 3$



Ex2: Find all real & cpx solutions of $z^4 - z^2 - 12 = 0$.

Let $y = z^2$, so that we get $y^2 - y - 12 = 0$. The latter quadratic eq-n has roots $y = 4$ and $y = -3$ (e.g. $y^2 - y - 12 = (y-4)(y+3)$ or use $y = \frac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-12)}}{2}$)

If $y = 4 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2$

If $y = -3 \Rightarrow z^2 = -3 \Rightarrow$ no real roots AND two complex roots $z = \pm\sqrt{3}\, i$

Answer: Real Solutions: $\{2, -2\}$, Complex solutions: $\{2, -2, \sqrt{3}\, i, -\sqrt{3}\, i\}$

Ex3: — " — $z^2 + 4z + 5 = 0$

Either use $z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

OR

$z^2 + 4z + 5 = (z+2)^2 + 1 \Rightarrow (z+2)^2 = -1 \Rightarrow z + 2 = \pm i \Rightarrow z = -2 \pm i$

Lecture #20 $\qquad$ §5.5 Complex Eigenvalues

The reason why we care about complex numbers is the following key result:

> **Claim**: Any polynomial $p(t)$ of degree n has exactly n roots, counting with multiplicities.

Example: For $p(t)=t^4+2t^2+1$, there are obviously no $\underbrace{\text{real roots}}_{\text{as } p(t)>0 \text{ for } t\in\mathbb{R}}$,
but $p(t)=(t^2+1)^2=(t-i)^2(t+i)^2 \Rightarrow$ it has complex roots i and $-i$, both with multiplicity 2 (and $2+2=4=\deg p(t)$).

In particular, we shall apply the previous eigenvalue/eigenvector theory for $\mathbb{C}^n$ (instead of $\mathbb{R}^n$), i.e. looking at $A\vec{x}=\lambda\vec{x}$ with $\lambda\in\mathbb{C}$, $\vec{x}\in\mathbb{C}^n$.

Basic Example: $A=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$ with $0<\varphi<\pi$.

$\det(A-\lambda I_2)=\lambda^2-2\cos\varphi\cdot\lambda+1 \leftarrow$ it has roots $\lambda=\cos\varphi\pm i\sin\varphi$

For $\lambda=\cos\varphi+i\sin\varphi$: $A-\lambda I_2=\begin{pmatrix}-i\sin\varphi & -\sin\varphi\\ \sin\varphi & -i\sin\varphi\end{pmatrix} : \begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}-i\sin\varphi\cdot x_1-\sin\varphi\cdot x_2\\ \sin\varphi\cdot x_1-i\sin\varphi\cdot x_2\end{pmatrix}$

$\Rightarrow$ e.g. $\vec{v_1}=\begin{pmatrix}1\\-i\end{pmatrix}$ - eigenvector

$\lambda=\cos\varphi-i\sin\varphi$: $A-\lambda I_2=\begin{pmatrix}i\sin\varphi & -\sin\varphi\\ \sin\varphi & i\sin\varphi\end{pmatrix} \Rightarrow$ e.g. $\vec{v_2}=\begin{pmatrix}1\\i\end{pmatrix}$ - eigenvect.

So: Eigenvalues of A are $\cos\varphi+i\sin\varphi$ and $\cos\varphi-i\cdot\sin\varphi$ with corresponding eigenvectors $\begin{pmatrix}1\\-i\end{pmatrix}$ and $\begin{pmatrix}1\\i\end{pmatrix}$.

Note: Knowing exact values of eigenvalues, the above search of eigenvectors boils down to finding a solution of only one linear eq-n! (compare to discussion in the bottom of p. 305 of the book)

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- Similar to $\operatorname{Re} x$ and $\operatorname{Im} x$ for $x \in \mathbb{C}$, we define $\underbrace{\operatorname{Re}\vec{x}}_{\boxed{\text{real part of } \vec{x}}}, \underbrace{\operatorname{Im}\vec{x}}_{\boxed{\text{imaginary part of } \vec{x}}} \in \mathbb{R}^n$ for $\vec{x} \in \mathbb{C}^n$, so that $\boxed{\vec{x} = \operatorname{Re}\vec{x} + i \cdot \operatorname{Im}\vec{x}}$

Example: $\operatorname{Re}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $\operatorname{Im}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, so that $\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}$

- For $A \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ (i.e. $m\times n$ matrix with complex entries), define $\overline{A} \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ by taking complex conjugate of each entry of A.

Properties:
1) $\overline{r\vec{x}} = \overline{r} \cdot \overline{\vec{x}}$ for $r \in \mathbb{C}$, $\vec{x} \in \mathbb{C}^n$
2) $\overline{B\vec{x}} = \overline{B} \cdot \overline{\vec{x}}$ for $B \in \mathrm{Mat}_{m\times n}(\mathbb{C})$, $\vec{x} \in \mathbb{C}^n$.
3) $\overline{BC} = \overline{B} \cdot \overline{C}$ for $B \in \mathrm{Mat}_{m\times n}(\mathbb{C})$, $C \in \mathrm{Mat}_{n\times k}(\mathbb{C})$
4) $\overline{rB} = \overline{r} \cdot \overline{B}$ for $r \in \mathbb{C}$, $B \in \mathrm{Mat}_{m\times n}(\mathbb{C})$

In particular, if $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, then $\overline{A\vec{x}} = \overline{A} \cdot \overline{\vec{x}} = A \cdot \overline{\vec{x}}$

Hence: $\boxed{\text{If } A\vec{x} = \lambda\vec{x}, \text{ then } A\overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}}$

This implies:

> If A is a real square matrix, its complex (non-real) eigenvalues occur in conjugate pairs
> i.e. those eigenvalues with $\operatorname{Im}\lambda \neq 0$

Example: In the above example of $A = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$, we indeed got

$$\lambda_2 = \cos\varphi - i\sin\varphi = \overline{\cos\varphi + i\sin\varphi} = \overline{\lambda_1}$$
$$\vec{v_2} = \begin{pmatrix}1\\i\end{pmatrix} = \overline{\begin{pmatrix}1\\-i\end{pmatrix}} = \overline{\vec{v_1}}$$

Ex 4: Find eigenvalues and eigenvectors for $A = \begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$

▸ $\lambda_1 = 1 + \sqrt{3}\cdot i$ $\quad \vec{v_1} = \begin{pmatrix}1\\-i\end{pmatrix}$

$\lambda_2 = 1 - \sqrt{3}\cdot i$ $\quad \vec{v_2} = \begin{pmatrix}1\\i\end{pmatrix}$

∎

Claim: Let $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and the corresponding eigenvector $\vec{v} \in \mathbb{C}^2$. Then:

$$A = PCP^{-1}, \text{ where } P = \begin{pmatrix}\operatorname{Re}\vec{v} & \operatorname{Im}\vec{v}\end{pmatrix} \text{ and } C = \begin{pmatrix}a & -b\\ b & a\end{pmatrix}$$

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Ex 5 (Exercise 27 from p. 310): Let A be an $n\times n$ real matrix such that $A^T = A$
i.e. A - symmetric.

Pick any $\vec{x}\in\mathbb{C}^n$ and set $q := \overline{\vec{x}}^T \cdot A \cdot \vec{x} \in \mathbb{C}$.

Show that q is a real number actually.

► $\overline{q} = \overline{\overline{\vec{x}}^T A \vec{x}} = \vec{x}^T \cdot \overline{A} \cdot \overline{\vec{x}} = \vec{x}^T \cdot A \cdot \overline{\vec{x}}$

$q = \overline{\vec{x}}^T \cdot A \cdot \vec{x} = \overline{\vec{x}}^T \cdot A^T \cdot \vec{x} = \left(\vec{x}^T \cdot A \cdot \overline{\vec{x}}\right)^T$

obvious as we take transpose of a 1x1 matrix ■

§5.7 Applications to Differential Equations

Finally, we shall discuss a very important application of eigenvalue/eigenvector problem to differential eq-s.

Setup: Solving a system of diff. equations:

$$\boxed{\begin{cases} x_1'(t) = a_{11}\cdot x_1(t) + \ldots + a_{1n}\cdot x_n(t) \\ \vdots \\ x_n'(t) = a_{n1}\cdot x_1(t) + \ldots + a_{nn}\cdot x_n(t) \end{cases}}$$

a_{ij} – constants $(1\le i,j\le n)$

$x_i(t)$ – differentiable functions

A nice way to rewrite this system is by using vector-valued function $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix}$, so that we get $\boxed{\vec{x}\,'(t) = A\cdot\vec{x}(t)}$ $(*)$

Clear: 1) $\vec{x}(t) = \vec{0}$ is a trivial solution

2) if $\vec{u}(t), \vec{v}(t)$ are solutions of $(*)$, so is any linear combination $c\cdot\vec{u}(t) + d\cdot\vec{v}(t)$.

General Result: The space of solutions of $(*)$ is n-dimensional

In what follows, we shall learn how to find a basis of this vector space.

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Simplest case: A - diagonal, so that $A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$

Then the system is decoupled (i.e. each equation involves only 1 f-n):

$$\begin{cases} x_1'(t) = a_{11} \cdot x_1(t) \\ x_2'(t) = a_{22} \cdot x_2(t) \\ \vdots \\ x_n'(t) = a_{nn} \cdot x_n(t) \end{cases} \Rightarrow \begin{cases} x_1(t) = C_1 \cdot e^{a_{11} t} \\ x_2(t) = C_2 \cdot e^{a_{22} t} \\ \vdots \\ x_n(t) = C_n \cdot e^{a_{nn} t} \end{cases}$$

So: $\boxed{\left\{ e^{a_{11} \cdot t} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e^{a_{22} \cdot t} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e^{a_{nn} \cdot t} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\} \text{ - basis of solutions}}$

Note: Each of the above functions is of the form $\vec{v} \cdot e^{\lambda t}$, where λ is an eigenvalue of A (diagonal above) and $\vec{v}$ is the corresponding eigenvector.

More generally, if A is an $n \times n$ real matrix and $\vec{v}$ is an eigenvector with eigenvalue λ, i.e. $A\vec{v} = \lambda \vec{v}$, then

$\boxed{\vec{x}(t) := e^{\lambda t} \cdot \vec{v} \text{ is a solution of } (*)}$

Indeed: $\vec{x}\,'(t) = \lambda \cdot e^{\lambda t} \cdot \vec{v} = e^{\lambda t} \cdot \lambda \vec{v} = e^{\lambda t} \cdot A\vec{v} = A \cdot \vec{x}(t)$.

Ex 6: Solve $\vec{x}\,'(t) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}(t)$ with $\vec{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.