Lecture #21

- Last time → Complex eigenvalues
  (If A is an $n\times n$ real matrix, then its complex (non-real) eigenvalues come in conjugate pairs)
  → We started the discussion of 1^{st} order differential systems

- **§5.7 Applications to Differential Equations**

Setup: Solving a system of n 1^{st} order differential equations in n differentiable functions $x_1(t), x_2(t), \ldots, x_n(t)$.

$$\begin{cases} x_1'(t) = a_{11} x_1(t) + \_\_\_ + a_{1n}\cdot x_n(t) \\ x_2'(t) = a_{21}\cdot x_1(t) + \_\_\_ + a_{2n}\cdot x_n(t) \\ \vdots \\ x_n'(t) = a_{n1}\cdot x_1(t) + \_\_\_ + a_{nn}\cdot x_n(t) \end{cases}$$

where a_{ij} are constants.

It is convenient to rewrite this system using vector-valued function $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, so that we get

$$\boxed{\vec{x}'(t) = A\cdot\vec{x}(t)} \quad (*)$$

Def: A solution of $(*)$ is a vector-valued function that satisfies $(*)$ for all t in a certain interval

Clear: 1) $\vec{x}(t) = \vec{0}$ is the trivial solution.

2) If $\vec{u}(t), \vec{v}(t)$ are solutions of $(*)$, so is any linear combin. $c\cdot\vec{u}(t) + d\cdot\vec{v}(t)$.

General Result from Differential Equations: The space of solutions of $(*)$ is n-dimensional.

However, if we fix a value at some point, e.g. $\vec{x}(0)$ the solution is unique.

This is the so called the initial value problem – i.e. finding a solution of $(*)$ satisfying $\vec{x}(0) = \vec{x_0}$, with $\vec{x_0}$ - given vector.

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• **Simplest Case**: A – diagonal, i.e. $A = \begin{pmatrix} a_{11} & & & & 0 \\ & a_{22} & & & \\ & & a_{33} & & \\ & & & \ddots & \\ 0 & & & & a_{nn} \end{pmatrix}$

Then, the system is decoupled (i.e. each equation involves only one function):

$$\begin{cases} x_1'(t) = a_{11} \cdot x_1(t) \\ \vdots \\ x_n'(t) = a_{nn} \cdot x_n(t) \end{cases} \Rightarrow \begin{cases} x_1(t) = C_1 \cdot e^{a_{11}t} \\ \vdots \\ x_n(t) = C_n \cdot e^{a_{nn}t} \end{cases} \Rightarrow \boxed{\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{a_{11}t} + \dots + C_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} e^{a_{nn}t}}.$$

So: $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{a_{11}t}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} e^{a_{22}t}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} e^{a_{nn}t} \right\}$ – basis of the solutions space.

Note: Each of these functions is of the form $\vec{v} \cdot e^{\lambda t}$, where λ is an eigenvalue of A (diagonal above) and $\vec{v}$ – corresponding eigenvector.

More generally, if A is an $n \times n$ matrix and $\vec{v}$ is an eigenvector of A with eigenvalue λ, i.e. $A\vec{v} = \lambda \cdot \vec{v}$, then

$$\boxed{\vec{x}(t) := \vec{v} \cdot e^{\lambda t} \text{ – a solution of } (*)}$$

[**Indeed**: $\vec{x}'(t) = \lambda e^{\lambda t} \vec{v} = e^{\lambda t} A\vec{v} = A\vec{x}(t)$]

Ex 1: Solve $\vec{x}'(t) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}(t)$ with $\vec{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

$\det(A - \lambda I_2) = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2) \Rightarrow$ eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 4$

$\lambda_1 = 2 \Rightarrow A - \lambda_1 I_2 = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \Rightarrow$ e.g. $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ – eigenvector

$\lambda_2 = 4 \Rightarrow A - \lambda_2 I_2 = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \Rightarrow$ e.g. $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ – eigenvector.

So: The general solution is $C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

Recover C_1, C_2 from $\vec{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ via $C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -2 \end{cases}$

Thus: $\vec{x}(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} - 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} = \begin{pmatrix} e^{2t} - 2e^{4t} \\ 3e^{2t} - 2e^{4t} \end{pmatrix}$

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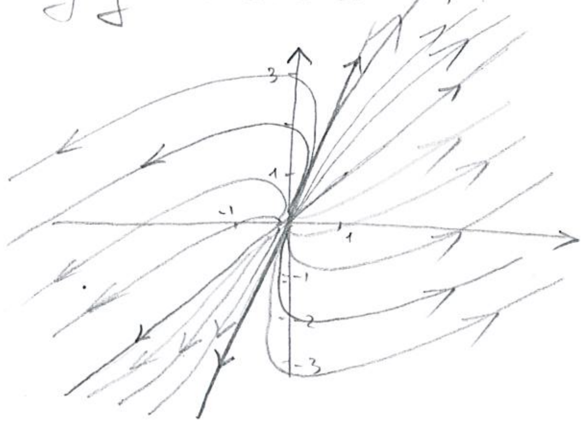
Let us look at the general solution from Ex1:

$$C_1 \cdot \begin{pmatrix}1\\3\end{pmatrix} e^{2t} + C_2 \begin{pmatrix}1\\1\end{pmatrix} e^{4t}$$

Varying constants C_1, C_2 we get all possible trajectories $\vec{x}(t)$



Note: 1) As $t \to \infty$, the term $C_2 \begin{pmatrix}1\\1\end{pmatrix} e^{4t}$ dominates (assuming $C_2 \neq 0$), hence it gets almost parallel to the line $y=x$ far away

2) As $t \to -\infty$, the term $C_1 \begin{pmatrix}1\\3\end{pmatrix} e^{2t}$ dominates (assuming $C_1 \neq 0$), hence, it gets almost parallel to the line $y=3x$ near origin

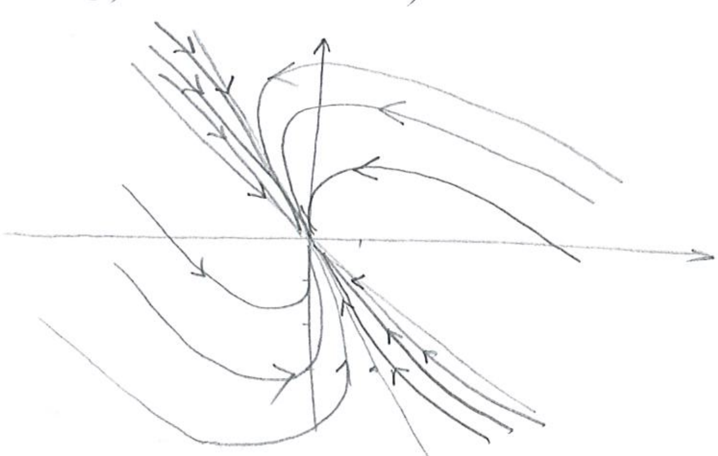
In this case (when both eigenvalues of 2×2 matrix A are positive numbers), the origin is called a repeller, or a source, of the dynamical system

! The direction of the greatest repulsion is the line corresponding to the eigenvector with the largest (of two) eigenvalue.

On the other hand, if we start from $A = \begin{pmatrix}-5 & -1\\ 3 & -1\end{pmatrix}$ instead of $\begin{pmatrix}5 & -1\\ 3 & 1\end{pmatrix}$, both eigenvalues are negative: $\lambda_1 = -2$ and $\lambda_2 = -4$, while the corresponding eigenvectors can be chosen as $\vec{v}_1 = \begin{pmatrix}1\\-3\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$.

In this case, the general solution has form

$C_1 \cdot \begin{pmatrix}1\\-3\end{pmatrix} \cdot e^{-2t} + C_2 \cdot \begin{pmatrix}1\\-1\end{pmatrix} e^{-4t}$, while the corresponding trajectories look as:



In this case, the origin is called an attractor or sink of dynamical system.

Direction of greatest attraction is the line corresponding to the eigenvector with the smallest eigenvalue.

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Let's now look at yet another pattern: when one eigenvalue is positive and another is negative.

Consider $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$.

Then: $\det(A - \lambda I_2) = (\lambda-3)(\lambda-1) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1) \Rightarrow$

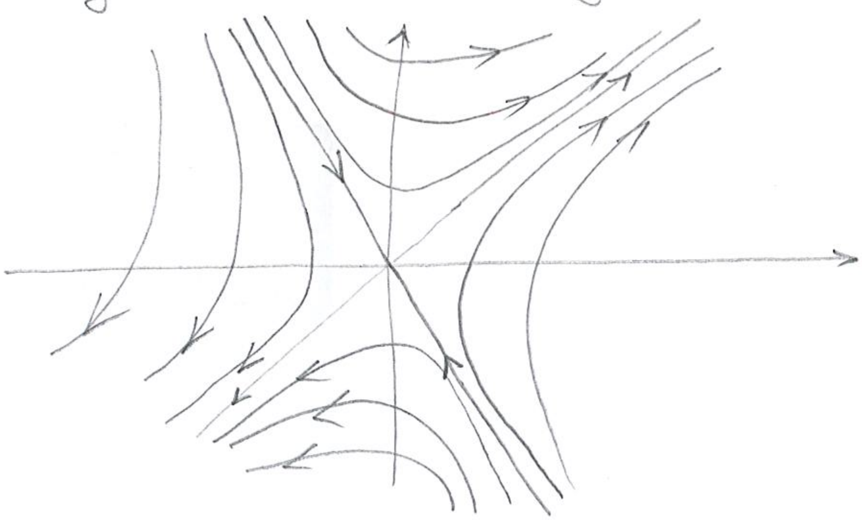
$\Rightarrow$ eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 5$.

Eigenvectors can be chosen as: $\vec{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus, the general solution is

$$C_1 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$$

The trajectories look as follows:



In this case (when one eigenvalue of 2×2 matrix) is positive and another is negative, the origin is called a saddle point of the dynamical system.

Note:
- The direction of the greatest repulsion is the line corresponding to the eigenvector with positive eigenvalue
- The direction of the greatest attraction is the line corresponding to the eigenvector with negative eigenvalue.

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Remark: If $n\times n$ matrix A is diagonalizable, i.e. $A = PDP^{-1}$ with diagonal D and $P = \begin{pmatrix} \vec{v_1} & \dots & \vec{v_n} \end{pmatrix}$ is obtained from eigenvectors, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

then switching from $\vec{x}(t)$ to $\vec{y}(t) := P^{-1}\cdot \vec{x}(t)$, the equation (*) reads

$$\boxed{\vec{y}\,'(t) = D\cdot \vec{y}(t)}$$, i.e. decoupled.

To obtain the general solution $\vec{x}(t)$, use $\vec{x}(t) = P\vec{y}(t)$

- Finally, we shall treat the case when a 2×2 matrix A does not have any real solutions (as we know it has a pair of conjugate complex eigenvalues)

Let's start from a general case, when $n\times n$ matrix A has a pair of conjugate complex eigenvalues $\lambda, \bar{\lambda}$ with corresponding complex eigenvectors $\vec{v}$ and $\bar{\vec{v}}$. In the original eq-n (*), all is real, so we expect to get a solution in terms of real numbers.

Instead of using $\vec{x}_1(t) = \vec{v}e^{\lambda t}$ and $\vec{x}_2(t) = \bar{\vec{v}}e^{\bar{\lambda} t} = \overline{\vec{x}_1(t)}$, we shall switch to

$$\boxed{\begin{aligned} \operatorname{Re} \vec{x_1}(t) &= \frac{1}{2}\left(\vec{x_1}(t) + \overline{\vec{x_1}(t)}\right) \\ \operatorname{Im} \vec{x_1}(t) &= \frac{1}{2i}\left(\vec{x_1}(t) - \overline{\vec{x_1}(t)}\right) \end{aligned}}$$

If $\lambda = a+bi$, then we shall use a nice formula:

$$e^{(a+bi)t} = e^{at}\left(\cos(bt) + i\cdot \sin(bt)\right)$$

<u>Then</u>: $\vec{x_1}(t) = (\operatorname{Re}\vec{v} + i\cdot \operatorname{Im}\vec{v})\cdot e^{at}\cdot(\cos(bt) + i\sin(bt))$

$$\Downarrow$$

$$\boxed{\begin{aligned} \vec{y_1}(t) &:= \operatorname{Re}\vec{x_1}(t) = [(\operatorname{Re}\vec{v})\cos bt - (\operatorname{Im}\vec{v})\sin bt]e^{at} \\ \vec{y_2}(t) &:= \operatorname{Im}\vec{x_1}(t) = [(\operatorname{Re}\vec{v})\sin bt + (\operatorname{Im}\vec{v})\cos bt]e^{at} \end{aligned}}$$ $\longleftarrow$ <u>So</u>: Instead of dealing with complex valued $\vec{x_1}(t)$, $\overline{\vec{x_1}}(t)$, we switch to $\vec{y_1}(t)$ and $\vec{y_2}(t)$.

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Consider $A = \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix}$.

$a=-2,\ b=1$

$\det(A - \lambda I_2) = \lambda^2 + 4\lambda + 5 \Rightarrow$ eigenvalues : $\lambda = -2+i$ and $\bar{\lambda} = -2-i$.

$\lambda = -2+i \Rightarrow A - \lambda I_2 = \begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix} \Rightarrow$ can take $\vec{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \Rightarrow \operatorname{Re}\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \operatorname{Im}\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

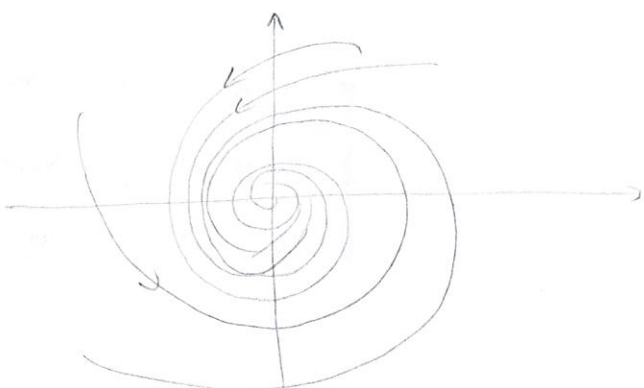
So:

$$\boxed{\begin{aligned} \vec{y_1}(t) &= \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right)e^{-2t} = \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix}e^{-2t} \\ \vec{y_2}(t) &= \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos t\right)e^{-2t} = \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}e^{-2t} \end{aligned}}$$

The general solution thus takes the form

$$C_1\begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}e^{-2t}$$

and the trajectories roughly look as:



The origin is called a spiral point of the dynamical system.

The spirals are oriented inward b/c of e^{-2t} (would be outward if we had e^{3t})

Suggested Reading: Practice Problem from p. 327 of the textbook

↑ its solution is presented on pp. 328-329.