Lecture #22

- § 4.1 → Vector spaces (formal definition ; examples: $\mathbb{R}^n$, $\mathbb{P}_n$, $Mat_{m\times n}$, $C(\mathbb{R})$)
  4.2 → Subspace of vector space (2 conditions to be checked!)
  ↳ Span of any collection of vectors is a subspace
  → linear combinations, spanning sets
  → Nul(A), Col(A), Row (A) and their explicit computation (+ bases for them)
  → Linear transformations $T: V \to W$; kernel & range of T.

§ 4.3 → Linearly independent sets (check just following definition)
→ Bases → the "Spanning Set Theorem" (p. 225 of the textbook)
↳ bases for Nul (A), Col (A), Row (A)

§ 4.5 → Dimension of a vector space
↳ how to determine bases & dim.
→ dim (subspace) ≤ dim (ambient vector space)
→ rank (A) + nullity (A) = # columns of A

- § 5.1 → Eigenvector & Eigenvalue & Eigenspace of a square matrix A
  ↳ simplest case : A is triangular
  → eigenvectors corresponding to distinct eigenvalues are lin. indep.

§ 5.2 → Characteristic equation & polynomial
↳ Find eigenvalues λ of A by solving $\det(A - \lambda \cdot I) = 0$
↳ for each of those find a basis of the null-space of $A - \lambda \cdot I$
→ Similar matrices A & B (if $A = PBP^{-1}$ for some P)
↳ they have the same char. pol-l, hence, same eigenvalues

§ 5.3 → Diagonalization : $A = PDP^{-1}$, D = diagonal
↳ diagonal entries of D are eigenvalues
→ columns of P are the corresponding eigenvectors
→ Application to the computation of A^n.

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- §5.4 → Eigenvectors & Eigenvalues of linear transf. $T: V \to V$
  - → The matrix of a linear transformation T relative to a basis $\mathcal{B}$: $[T]_{\mathcal{B}}$ ← need to know how to compute
  - → If $A = PBP^{-1}$ and $\mathcal{B}$ is a basis for $\mathbb{R}^n$ formed by columns of P, then $[T]_{\mathcal{B}} = B$, where $T: \vec{x} \mapsto A\vec{x}$

- Appendix B → Complex numbers $\mathbb{C}$ (see in-class & textbook discussion)
  (division via conjugate / polar form $re^{i\theta}$ / de Moivre's Theorem)

- §5.5 → Complex eigenvalues
  - → for real matrices, the cpx non-real eigenvalues come in conjugate pairs
  - → Theorem 9 on p. 309 of the textbook!
    $$A = P \cdot C \cdot P^{-1}, \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad P = \begin{pmatrix} \mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v} \end{pmatrix}$$
    given that $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ with eigenvalue $\lambda = \underset{\mathbb{R}}{a} - \underset{\mathbb{R}}{b}i \in \mathbb{C}$
    and eigenvector $\vec{v} \in \mathbb{C}^2$
  - → Review the formula for the roots of a quadratic polynomial (as well as how we find roots of cubic polynomials via factorization)

- §5.7 → Solving $\vec{x}\,'(t) = A \cdot \vec{x}(t)$ with $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$
  - → for $A \in \mathrm{Mat}_{2\times 2}$, must be able to determine if the origin $(0,0)$ is an attractor = sink, or repeller = source, or saddle point, or a spiral point
  - → if $\lambda_1, \ldots, \lambda_n$ are real eigenvalues of A with eigenbasis $\vec{v_1}, \ldots, \vec{v_n}$, then <u>general solution</u> is $\boxed{C_1 e^{\lambda_1 t}\vec{v_1} + C_2 e^{\lambda_2 t}\vec{v_2} + \ldots + C_n e^{\lambda_n t}\vec{v_n}}$
    - → determine $C_1, \ldots, C_n$ explicitly for <u>initial value problem</u>
  - → if $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ has a complex eigenvalue $\lambda = a+bi$ (a,b - real, $b \neq 0$) with eigenvector $\vec{v} \in \mathbb{C}^2$ (so $\vec{v} = \mathrm{Re}\,\vec{v} + i\cdot \mathrm{Im}\,\vec{v}$ with $\mathrm{Re}(\vec{v}), \mathrm{Im}(\vec{v}) \in \mathbb{R}^2$), then

<u>general solution</u>: $\boxed{C_1 [\mathrm{Re}\,\vec{v}\cdot\cos(bt) - \mathrm{Im}\,\vec{v}\cdot\sin(bt)]e^{at} + C_2 [\mathrm{Re}\,\vec{v}\cdot\sin bt + \mathrm{Im}\,\vec{v}\cdot\cos bt]e^{at}}$