Lecture #23

§6.1 Inner Product, Length, Angles, Orthogonality

We shall see now how familiar concepts from $\mathbb{R}^2$ (plane) or $\mathbb{R}^3$ (space) can be generalized to an arbitrary $\mathbb{R}^n$.

Def: Given two vectors $\vec{u}=\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\vec{v}=\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$, the inner product (a.k.a. the dot product), denoted $\vec{u}\cdot\vec{v}$, is the number $\vec{u}^T\cdot\vec{v}$

Here: $\vec{u}^T$ is an $1\times n$ matrix, $\vec{v}$ is an $n\times 1$ matrix $\Rightarrow$ product $\vec{u}^T\cdot\vec{v}$ is a 1×1 matrix, i.e. a number.

Ex1: Compute the inner product of $\vec{u}=\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} 5 \\ -3 \\ 1 \\ -2 \end{pmatrix}$ in $\mathbb{R}^4$.

▶ $\vec{u}\cdot\vec{v} = 1\cdot 5 + 2\cdot(-3) + 3\cdot 1 + 4\cdot(-2) = -6$ ∎

Properties of the inner product

1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
3) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$
4) $\vec{u}\cdot\vec{u} \geq 0$ (with equality iff $\vec{u}=0$)

for any $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^n$, $c\in\mathbb{R}$

If $\vec{u}=\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, then $\vec{u}\cdot\vec{u} = u_1^2+u_2^2+\_\_+u_n^2$.

Def: Given a vector $\vec{v}=\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$, the length (a.k.a. the norm) of $\vec{v}$, denoted $\|\vec{v}\|$, is defined via $\|\vec{v}\| := \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2+v_2^2+\_\_+v_n^2} \in \mathbb{R}_{\geq 0}$

Rmk: For $n=2,3$, this coincides with the usual notion of length, due to Pythagorean Thm.

Ex2: Find $\|\vec{u}\|$ and $\|\vec{v}\|$ in the setup of Ex1

▶ $\|\vec{u}\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$

$\|\vec{v}\| = \sqrt{5^2+(-3)^2+1^2+(-2)^2} = \sqrt{39}$ ∎

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Properties: 1) $\| c \cdot \vec{u} \| = \| \vec{u} \| \cdot \underbrace{|c|}_{\text{absolute value of } c}$

2) $\| \vec{u} + \vec{v} \| \le \| \vec{u} \| + \| \vec{v} \|$ (the triangle inequality)

Def: A vector $\vec{v}$ is called a unit vector iff $\| \vec{v} \| = 1$.

! Given a nonzero vector $\vec{v} \in \mathbb{R}^n$, a unit vector in the direction of $\vec{v}$ is given by $\vec{u} := \frac{1}{\| \vec{v} \|} \cdot \vec{v}$, while a unit vector in the opposite direction is $\vec{w} = \frac{-1}{\| \vec{v} \|} \cdot \vec{v}$

Ex 3: Find a unit vector in the direction of $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \in \mathbb{R}^2$

$\vec{u} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$ ∎

Def: Given a pair of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the distance b/w $\vec{u}$ and $\vec{v}$, denoted $\text{dist}(\vec{u}, \vec{v})$, is the length of $\vec{u} - \vec{v}$, i.e.

$$\text{dist}(\vec{u}, \vec{v}) = \| \vec{u} - \vec{v} \|$$

Ex 4: Compute $\text{dist}(\vec{u}, \vec{v})$ in the setup of Ex 1

$\text{dist}(\vec{u}, \vec{v}) = \sqrt{(-4)^2 + 5^2 + 2^2 + 6^2} = \sqrt{81} = 9$ ∎

Def: Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal iff $\vec{u} \cdot \vec{v} = 0$

Claim: For two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, we have
$\vec{u} \cdot \vec{v} = \| \vec{u} \| \cdot \| \vec{v} \| \cdot \cos\theta$, where θ – angle b/w $\vec{u}$ and $\vec{v}$

! In particular, two nonzero vectors in $\mathbb{R}^n$ are orthogonal iff the angle b/w them is $90° = \pi/2$

Note: $\vec{0}$ is orthogonal to any $\vec{u} \in \mathbb{R}^n$

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Claim: Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal iff $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

[It's instructive to present a proof of this claim:

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \underbrace{\vec{u}\cdot\vec{u}}_{\|\vec{u}\|^2} + \underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2} + \underbrace{(\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u})}_{2\vec{u}\cdot\vec{v}} \Rightarrow \text{Claim follows}$$
]

The formula $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$ (θ - angle b/w $\vec{u}, \vec{v}$) is often used to determine the angle between two vectors

Ex5: Find the angle θ b/w two vectors in Ex1.

By Ex1, Ex2, have:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{-6}{\sqrt{30}\cdot\sqrt{39}} = -\frac{6}{3\cdot\sqrt{130}} = -\frac{2}{\sqrt{130}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-2/\sqrt{130}\right)$$

Def: For a subspace W of $\mathbb{R}^n$, its orthogonal complement, denoted $W^\perp$, is the set of all $\vec{z} \in \mathbb{R}^n$ which are orthogonal to any $\vec{w} \in W$:

$$W^\perp = \{\vec{z} \in \mathbb{R}^n \mid \vec{z}\cdot\vec{w} = 0 \text{ for any } \vec{w} \in W\}$$

Q: If $W = \{\vec{0}\} \subseteq \mathbb{R}^n$, what is $W^\perp$?

A: $W^\perp = \mathbb{R}^n$

Q: If $W = \mathbb{R}^n$, what is $W^\perp$?

A: $W^\perp = \{\vec{0}\}$

Q: If $W \subseteq \mathbb{R}^2$ is a subspace consisting of all vectors lying on a given line ℓ through the origin, describe $W^\perp$.

A: $W^\perp$ is a subspace consisting of all vectors lying on an orthogonal line $\ell^\perp$ through the origin.

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Let A be an $m \times n$ matrix. Recall the following subspaces (discussed before):

Col A – subspace of $\mathbb{R}^m$

Row A – subspace of $\mathbb{R}^n$

Nul A – subspace of $\mathbb{R}^n$

$$\boxed{\text{Claim: } (\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T}$$

! Discuss a straightforward proof

Properties of $W^\perp$

1) $W^\perp$ is a subspace

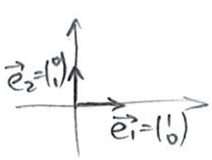
2) If W is spanned by $\vec{w_1}, \ldots, \vec{w_k} \in \mathbb{R}^n$, then $W^\perp$ consists of all $\vec{z} \in \mathbb{R}^n$ orthogonal to each $\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}$

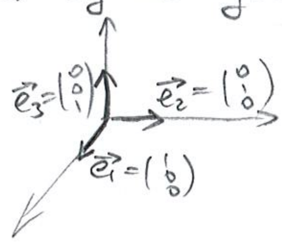
§6.2 Orthogonal Sets

Def: A set of vectors $S = \{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if any two distinct vectors in S are orthogonal:

$$\vec{u_i} \cdot \vec{u_j} = 0 \text{ for any } i \neq j$$

Examples:





Claim: If $S = \{\vec{u_1}, \ldots, \vec{u_k}\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.

Again, it is instructive to sketch the proof:

If $c_1\vec{u_1} + c_2\vec{u_2} + \ldots + c_k\vec{u_k} = \vec{0}$, then taking inner product with $\vec{u_1}$ we get $c_1 \cdot (\vec{u_1} \cdot \vec{u_1}) + 0 + \ldots + 0 = 0 \Rightarrow c_1 = 0$

likewise: $c_2 = 0, \ldots, c_k = 0$

$\Big\} \Rightarrow$ all weights $= 0$ $\Rightarrow S$ – lin. indep.

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Def: An *orthogonal basis* for a subspace $W \subseteq \mathbb{R}^n$ is a basis of W, which is also an orthogonal set.

Rmk: Due to the previous claim, orthogonal basis for W is an orthogonal set of elements of W, spanning W.

Claim: Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\vec{y}$ in W, the weights in the linear combination

$$\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_k\vec{u}_k$$

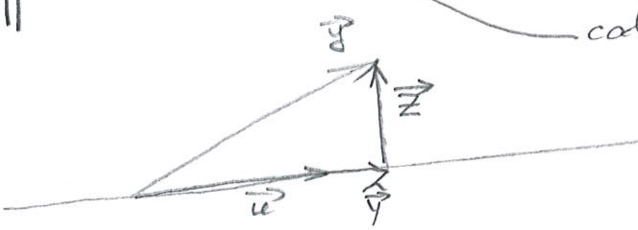
are given by

$$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j} = \frac{\vec{y}\cdot\vec{u}_j}{\|\vec{u}_j\|^2}$$

Def: Given a nonzero vector $\vec{u}$ in $\mathbb{R}^n$, an *orthogonal projection* of a vector $\vec{y} \in \mathbb{R}^n$ onto $\vec{u}$ is the vector $\hat{\vec{y}} \in \mathbb{R}^n$ such that

1) $\hat{\vec{y}}$ is a multiple of $\vec{u}$
2) $\vec{z} := \vec{y} - \hat{\vec{y}}$ is orthogonal to $\vec{u}$

called *component of $\vec{y}$ orthogonal to $\vec{u}$*



From 1), we see that $\hat{\vec{y}} = \alpha\cdot\vec{u}$ for some constant α.

Taking inner product of $\vec{y} = \alpha\vec{u} + \vec{z}$ with $\vec{u}$, we get

$$\vec{y}\cdot\vec{u} = \alpha\cdot\vec{u}\cdot\vec{u} + 0 \Rightarrow \alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}.$$

So: $$\boxed{\hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\cdot\vec{u}}$$

Note: The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is the same as an any $c\cdot\vec{u}$, hence, the notation $\text{proj}_L\vec{y}$ for $\hat{\vec{y}}$, where L - subspace spanned by $\vec{u}$.

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Ex6: Verify that $\vec{u_1} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\vec{u_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{u_3} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$ is an orthogonal basis of $\mathbb{R}^3$

▶ Verify $\vec{u_1} \cdot \vec{u_2} = 0$, $\vec{u_1} \cdot \vec{u_3} = 0$, $\vec{u_2} \cdot \vec{u_3} = 0$.

As they are lin. independent (By Claim) and there 3 of them of height 3, they must be a basis

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Def: (1) $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ is an orthonormal set of $\mathbb{R}^n$ iff it is an orthogonal set and each vector is a unit vector

($i.e.$ $\vec{u_i} \cdot \vec{u_j} = 0$ for $i \neq j$ AND $\vec{u_i} \cdot \vec{u_i} = 1$)

(2) An orthonormal basis of a subspace W is a basis of W that is also an orthonormal set

Ex7: Construct an orthonormal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ of $\mathbb{R}^3$ with $\vec{v_1}$ in the direction of $\vec{u_1}$ from Ex6.

▶ $\vec{v_1} = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\vec{v_2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v_3} = \frac{1}{\sqrt{42}} \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

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