Lecture #26

- Last time → Gram-Schmidt process (to find an orthogonal basis of a subspace of $\mathbb{R}^n$)
  - → QR-factorization
    (recover Q from Gram-Schmidt; recover R from $R = Q^T A$)
  - → started the least-squares problems

- Finish pages 5-7 of Lecture 25 notes on the least-squares solutions

- §6.7 Inner Product Spaces

For the rest of today we are going to discuss how all the previous concepts (length, distance, orthogonality, orthogonal bases) can be generalized when $\mathbb{R}^n$ is replaced by a vector space V.

The starting point is the following definition:

Def: An inner product on a vector space V is a function $V \times V \xrightarrow{\langle \cdot , \cdot \rangle} \mathbb{R}$ that satisfies the following axioms:

1) $\langle u, v\rangle = \langle v, u\rangle$
2) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
3) $\langle cu, v\rangle = c\cdot\langle u, v\rangle$
4) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$

A vector space with an inner product is called an inner product space

Key Example: $V = \mathbb{R}^n$ with the standard inner (= dot) product

Ex1: a) Show that $\langle \vec{u}, \vec{v}\rangle = u_1v_1 + 3u_2v_2 + 5u_3v_3$ defines an inner product on $\mathbb{R}^3$

b) Show that $\langle \vec{u}, \vec{v}\rangle = u_1v_1 - 3u_2v_2 - 5u_3v_3$ does not define an inner product on $\mathbb{R}^3$

(Here, we write $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$)

Lecture #26

Ex 2: Let $t_0, t_1, \dots, t_m$ be distinct real numbers.

Let $V = \mathbb{P}_n$ and define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ via

$$\langle p(t), q(t) \rangle := p(t_0)q(t_0) + p(t_1)q(t_1) + \dots + p(t_m)q(t_m).$$

a) Show that for $m \geq n$, $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{P}_n$

b) Show that for $m < n$, $\langle \cdot, \cdot \rangle$ does not define an inner product on $\mathbb{P}_n$.

Ex 3: Let $V = C[a,b]$ be the vector space of all continuous functions on the interval $[a,b]$. Show that

$$\langle f(t), g(t) \rangle := \int_a^b f(t)g(t)\,dt$$

defines an inner product on V.

Note: $C[a,b]$ is HUGE! (i.e. it is not finite dimensional)

Now, given an inner product space V, one can apply all the discussions from the previous 3 classes. In particular:

(1) the length (or norm) of $\vec{v} \in V$ is defined as

$$\boxed{\|\vec{v}\| := \sqrt{\langle \vec{v}, \vec{v} \rangle}}$$

2) $\vec{v} \in V$ is unit iff $\boxed{\|\vec{v}\| = 1}$

3) the distance between $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$

4) $\vec{u}, \vec{v} \in V$ are orthogonal iff $\boxed{\langle \vec{u}, \vec{v} \rangle = 0}$

! We can also:

5) apply Gram-Schmidt process to find an orthogonal basis

6) compute projection of $\vec{v}$ onto a subspace $W \subseteq V$

7) find best approximation

Lecture #26

Ex 4: Let $V = \mathbb{P}_2$ and choose $t_0 = 0,\ t_1 = 1,\ t_2 = 2$.

Let $p(t) = t+1$, $q(t) = t^2 + t + 1$.

1) Find $\langle p, q \rangle$ and $\langle q, p \rangle$

2) Find lengths of $p(t)$ and $q(t)$.

1) $\langle p, q \rangle = (t+1)|_{t=0} \cdot (t^2+t+1)|_{t=0} + (t+1)|_{t=1} (t^2+t+1)|_{t=1} + (t+1)|_{t=2} (t^2+t+1)|_{t=2}$

$= 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 7 = \underline{\underline{28}}$

$\langle q, p \rangle = \langle p, q \rangle = 28$

2) $\langle p, p \rangle = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14 \Rightarrow \| p(t) \| = \sqrt{14}$

$\langle q, q \rangle = 1 \cdot 1 + 3 \cdot 3 + 7 \cdot 7 = 59 \Rightarrow \| q(t) \| = \sqrt{59}$

Ex 5: Let $V = C[0,2]$ with the inner product of Ex 3.

Let $p(t) = t+1$, $q(t) = t^2+t+1$.

1) Find $\langle p, q \rangle$ and $\langle q, p \rangle$

2) Find lengths of $p(t)$ and $q(t)$.

1) $\langle p, q \rangle = \int_0^2 (t+1)(t^2+t+1)\,dt = \int_0^2 (t^3 + 2t^2 + 2t + 1)\,dt = \left(\frac{t^4}{4} + \frac{2}{3}t^3 + t^2 + t \right) \Big|_{t=0}^{t=2}$

$= \left(4 + \frac{16}{3} + 4 + 2\right) - 0 = \frac{46}{3}$

$\langle q, p \rangle = \langle p, q \rangle = \frac{46}{3}$

2) $\langle p, p \rangle = \int_0^2 (t+1)^2\,dt = \int_0^2 (t^2 + 2t + 1)\,dt = \left(\frac{t^3}{3} + t^2 + t \right) \Big|_{t=0}^{t=2}$

$= \frac{8}{3} + 6 = \frac{26}{3}$

$\Rightarrow \| p(t) \| = \sqrt{\frac{26}{3}}$

$\langle q, q \rangle = \int_0^2 (t^2+t+1)^2\,dt = \int_0^2 (t^4 + t^2 + 1 + 2t^3 + 2t^2 + 2t)\,dt$

$= \left(\frac{t^5}{5} + \frac{1}{2}t^4 + t^3 + t^2 + t \right) \Big|_{t=0}^{t=2} = \frac{32}{5} + 8 + 8 + 4 + 2 = \frac{142}{5}$

$\Rightarrow \| q(t) \| = \sqrt{\frac{142}{5}}$

Lecture #26

Ex 6: Let $V = C[0,2]$ be the inner product space from Ex 5.
Find an orthogonal basis of the subspace $W \subset V$ spanned by
$p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$.

▸ Set $q_1(t) := p_1(t) = 1$ (in our old notations, $p_1(t)$ is $\vec{x_1}$, $q_1(t)$ is $\vec{v_1}$)

Set $q_2(t) := p_2(t) - \dfrac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} \cdot q_1(t)$

$\langle p_2, q_1\rangle = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_{t=0}^{t=2} = 2$

$\langle q_1, q_1\rangle = \int_0^2 1\,dt = 2$

$\Rightarrow q_2(t) = t - 1$

Set $q_3(t) := p_3(t) - \dfrac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1(t) - \dfrac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2(t)$

$\langle p_3, q_1\rangle = \int_0^2 t^2\,dt = \frac{t^3}{3}\Big|_{t=0}^{t=2} = \frac{8}{3}$

$\langle q_1, q_1\rangle = 2$

$\langle p_3, q_2\rangle = \int_0^2 (t^3 - t^2)\,dt = \left(\frac{t^4}{4} - \frac{t^3}{3}\right)\Big|_{t=0}^{t=2} = 4 - \frac{8}{3} = \frac{4}{3}$

$\langle q_2, q_2\rangle = \int_0^2 (t-1)^2\,dt = \int_0^2 (t^2 - 2t + 1)\,dt = \left(\frac{t^3}{3} - t^2 + t\right)\Big|_{t=0}^{t=2} = \frac{8}{3} - 4 + 2 = \frac{2}{3}$

$$\Rightarrow q_3(t) = t^2 - \frac{4}{3}\cdot 1 - 2(t-1) = t^2 - 2t + \frac{2}{3}$$

Thus:
$$\boxed{\begin{aligned} q_1(t) &= 1 \\ q_2(t) &= t - 1 \\ q_3(t) &= t^2 - 2t + \frac{2}{3} \end{aligned}}$$
– orthogonal basis of W ∎

Note: Here, we applied the Gram-Schmidt algorithm on the nose!

Note: Once you have an orthogonal basis of $W \subseteq V$, one can compute $\text{proj}_W \vec{v}$ in exactly the same way as before.

Lecture #26

Def: Given a subspace W of the inner product space V and an element $\vec{v} \in V$, the best approximation to $\vec{v}$ by elements in W is the element $\vec{w} \in W$ such that $\|\vec{v} - \vec{w}\|$ – the smallest possible

Obviously: $\vec{w} = \text{proj}_W \vec{v}$ – the orthogonal projection of $\vec{v}$ onto W.

Ex 7: In the setup of Ex 6, find the best approximation of $p(t) = t^3$ by the elements in $W = \text{span}\langle 1, t, t^2 \rangle$.

▸ $\vec{w} = \text{proj}_W p$

Recall the orthogonal basis of W from Ex 6:

$$q_1(t) = 1, \quad q_2(t) = t - 1, \quad q_3(t) = t^2 - 2t + \frac{2}{3}.$$

Then: $\text{proj}_W p = \frac{\langle p, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1(t) + \frac{\langle p, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2(t) + \frac{\langle p, q_3 \rangle}{\langle q_3, q_3 \rangle} q_3(t)$

$$\left.\begin{aligned} \langle p, q_1 \rangle &= \int_0^2 t^3 \, dt = \frac{t^4}{4}\Big|_{t=0}^{t=2} = 4 \\ \langle q_1, q_1 \rangle &= 2 \end{aligned}\right\} \Rightarrow \frac{\langle p, q_1 \rangle}{\langle q_1, q_1 \rangle} = 2$$

$$\left.\begin{aligned} \langle p, q_2 \rangle &= \int_0^2 (t^4 - t^3) \, dt = \left(\frac{t^5}{5} - \frac{t^4}{4}\right)\Big|_{t=0}^{t=2} = \frac{32}{5} - 4 = \frac{12}{5} \\ \langle q_2, q_2 \rangle &= \frac{2}{3} \end{aligned}\right\} \Rightarrow \frac{\langle p, q_2 \rangle}{\langle q_2, q_2 \rangle} = \frac{18}{5}$$

$$\left.\begin{aligned} \langle p, q_3 \rangle &= \int_0^2 \left(t^5 - 2t^4 + \frac{2}{3}t^3\right) dt = \left(\frac{t^6}{6} - \frac{2}{5}t^5 + \frac{1}{6}t^4\right)\Big|_{t=0}^{t=2} = \frac{64}{6} - \frac{64}{5} + \frac{16}{6} \\ &= \frac{80}{6} - \frac{64}{5} = \frac{400 - 384}{30} = \frac{16}{30} = \frac{8}{15} \\ \langle q_3, q_3 \rangle &= \int_0^2 \left(t^2 - 2t + \frac{2}{3}\right)^2 dt = \int_0^2 \left(t^4 + 4t^2 + \frac{4}{9} - 4t^3 + \frac{4}{3}t^2 - \frac{8}{3}t\right) dt \\ &= \left(\frac{t^5}{5} - t^4 + \frac{16}{9}t^3 - \frac{4}{3}t^2 + \frac{4}{9}t\right)\Big|_{t=0}^{t=2} = \frac{32}{5} - 16 + \frac{128}{9} - \frac{16}{3} + \frac{8}{9} \\ &= -\frac{48}{5} + \frac{88}{9} = \frac{440 - 432}{45} = \frac{8}{45} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \frac{\langle p, q_3 \rangle}{\langle q_3, q_3 \rangle} = \frac{8}{15} \cdot \frac{45}{8} = 3$$

So: $\vec{w} = \text{proj}_W p = 2 \cdot 1 + \frac{18}{5}(t - 1) + 3\left(t^2 - 2t + \frac{2}{3}\right)$ ◼

Lecture # 26

We conclude this class by the following two useful inequalities

Claim (Cauchy-Schwarz inequality): Given an inner product space V:

$$|\langle \vec{u}, \vec{v} \rangle| \le \|\vec{u}\| \cdot \|\vec{v}\| \quad \text{for any } \vec{u}, \vec{v} \in V.$$

Claim (Triangle inequality): Given an inner product space V:

$$\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\| \quad \text{for any } \vec{u}, \vec{v} \in V$$

See page 403 of the textbook for the proofs of both results.

Remark: The Cauchy-Schwarz inequality allows to define an angle θ between two vectors $\vec{u}$ & $\vec{v}$ in any inner product space V via:

$$\boxed{\theta = \cos^{-1}\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|} \right)}$$