Lecture # 27

- Last time $\to$ Inner product spaces
  - $\to$ Recall 4 axioms
  - $\to$ Recall two examples with $V = \mathbb{P}_n$ or $C[a,b]$.
  - $\to$ Can apply all results for $\mathbb{R}^n$ from Section 6!

- §7.1 Diagonalization of symmetric matrices

Recall that a matrix A is symmetric iff $A^T = A$

(in particular, if $A \in \mathrm{Mat}_{m\times n} \Rightarrow A^T \in \mathrm{Mat}_{n\times m} \Rightarrow m = n$).

Ex1: Let $\mathrm{Mat}^{sym}_{n\times n}$ denote the subset of $\mathrm{Mat}_{n\times n}$ consisting of symmetric ones

1) Verify that $\mathrm{Mat}^{sym}_{n\times n}$ is a subspace of $\mathrm{Mat}_{n\times n}$
2) Find a basis of $\mathrm{Mat}^{sym}_{n\times n}$
3) Compute $\dim \mathrm{Mat}^{sym}_{n\times n}$

Ex2: Diagonalize (if possible) symmetric matrix $A = \begin{pmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{pmatrix}$

$$\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -6 & 4 \\ -6 & 2-\lambda & -2 \\ 4 & -2 & 3-\lambda \end{vmatrix} = (1-\lambda)\underbrace{\begin{vmatrix} 2-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix}}_{-10+\lambda+\lambda^2} + 6\underbrace{\begin{vmatrix} -6 & -2 \\ 4 & 3-\lambda \end{vmatrix}}_{6\lambda+26} + 4\underbrace{\begin{vmatrix} -6 & 2-\lambda \\ 4 & -2 \end{vmatrix}}_{4+4\lambda}$$

$$= -\lambda^3 + 63\lambda + 162 = -(\lambda^3 - 63\lambda - 162)$$

Note: -3 is a root of this polynomial as $-27 + 189 - 162 = 0$

$\Rightarrow \lambda + 3$ divides $\lambda^3 - 63\lambda - 162$ and explicitly get:

$$\lambda^3 - 63\lambda - 162 = (\lambda+3)(\lambda^2 - 3\lambda - 54) = (\lambda+3)(\lambda+6)(\lambda-9)$$

So: eigenvalues are $\lambda = -3, -6, 9$

Lecture #27

▸ (Continuation of Ex 2)

As there are 3 distinct eigenvalues, each corresponding eigenspace is 1-dim and we need to choose nonzero el-s in each.

$$\lambda=-3 \Rightarrow A+3I_3=\begin{pmatrix} 4 & -6 & 4 \\ -6 & 5 & -2 \\ 4 & -2 & 0 \end{pmatrix} \Rightarrow \text{choose } \vec{v_1}=\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda=-6 \Rightarrow A+6I_3=\begin{pmatrix} 7 & -6 & 4 \\ -6 & 8 & -2 \\ 4 & -2 & 3 \end{pmatrix} \Rightarrow \text{choose } \vec{v_2}=\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda=9 \Rightarrow A-9I_3=\begin{pmatrix} -8 & -6 & 4 \\ -6 & -7 & -2 \\ 4 & -2 & -12 \end{pmatrix} \Rightarrow \text{choose } \vec{v_3}=\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

So: $A=P\cdot D\cdot P^{-1}$ with $P=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ and $D=\begin{pmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ ■

BUT: the matrix P from the above proof has a nice property, namely, its columns form an orthogonal set (hence, an orthogonal basis) of $\mathbb{R}^3$. Indeed

$$\vec{v_1}\cdot\vec{v_2}=\vec{v_1}\cdot\vec{v_3}=\vec{v_2}\cdot\vec{v_3}=0.$$

From previous discussions we know that sometimes it's useful to have an orthonormal rather than orthogonal basis

So: choose $\vec{u_1}:=\dfrac{\vec{v_1}}{\|\vec{v_1}\|}=\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$, $\vec{u_2}=\dfrac{\vec{v_2}}{\|\vec{v_2}\|}=\begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$, $\vec{u_3}=\dfrac{\vec{v_3}}{\|\vec{v_3}\|}=\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$

Then: $\boxed{A=P\cdot D\cdot P^{-1} \text{ with } D=\begin{pmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{pmatrix} \text{ and } P=\begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}}$

$\quad \| \,!\, P$-orthogonal

$\quad P\cdot D\cdot P^{T}$

Lecture #27

Def: An $n \times n$ matrix A is said to be orthogonally diagonalizable if there is a diagonal matrix D and an orthogonal matrix P:

$$\boxed{A = PDP^{-1} = PDP^{T}}$$

P-orthogonal $\Leftrightarrow P^{-1} = P^{T}$

Rmk: A-orthogonally diagonalizable $\underset{\not\Leftarrow}{\Rightarrow}$ A-diagonalizable.

Main Claim: An $n \times n$ matrix A is orthogonally diagonalizable iff A-symmetric

Note: the direction "$\Rightarrow$" is easy:

if $A = PDP^{T} \Rightarrow A^{T} = (PDP^{T})^{T} = (P^{T})^{T} \cdot D^{T} \cdot P^{T} = P \cdot D \cdot P^{T} = A$

The orthogonality of $\vec{v_1}, \vec{v_2}, \vec{v_3}$ from the proof of Ex 2 is not just a mere luck as the following result says:

Claim: If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal

see p.420 for a proof.

The above results may be summarized as follows:

Claim (Spectral theorem for symmetric matrices):

Let A be a symmetric $n \times n$ matrix. Then:

1) A has n real eigenvalues, counting multiplicities
2) The dimension of the eigenspace for each eigenvalue λ equals the algebraic multiplicity of λ
3) The eigenspaces are mutually orthogonal
4) A is orthogonally diagonalizable.

Let's conclude by obtaining another formula for A as above.

Let $A = PDP^{-1} = PDP^T$ with $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $P = \begin{pmatrix} \vec{u_1} & \cdots & \vec{u_n} \end{pmatrix}$ ← orthogonal

So $\{\vec{u_1}, \ldots, \vec{u_n}\}$ – orthonormal basis of $\mathbb{R}^n$.

Then:
$$A = \begin{pmatrix} \vec{u_1} & \cdots & \vec{u_n} \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \cdot \begin{pmatrix} \vec{u_1}^T \\ \vdots \\ \vec{u_n}^T \end{pmatrix}$$

$\Downarrow$

$$\boxed{A = \lambda_1 \cdot \vec{u_1} \cdot \vec{u_1}^T + \cdots + \lambda_n \cdot \vec{u_n} \cdot \vec{u_n}^T}$$ ← spectral decomposition of A

Here: $\vec{u_i}$ is an $n \times 1$ matrix, $\vec{u_i}^T$ – an $1 \times n$ matrix $\Rightarrow \vec{u_i} \cdot \vec{u_i}^T$ – an $n \times n$ matrix

Q: What is the rank of $\vec{u_i}\vec{u_i}^T$?

A: $rk = 1$.

Lecture #27

Ex 3: Orthogonally diagonalize $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

Step 1: Find eigenvalues

$$\det(A - \lambda I_3) = \dots = -(\lambda^3 - 6\lambda^2 + 9\lambda - 4) = -(\lambda-1)^2(\lambda-4)$$

Step 2: Eigenspaces

$\lambda = 4 \rightsquigarrow \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ Null is spanned by $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda = 1 \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ Null is spanned by

$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

But: We need an orthonormal eigenbasis

Step 3: Apply Gram-Schmidt to the $\lambda = 1$ eigenspace, i.e.
replace $\vec{v}_3$ by $\vec{w} = \vec{v}_3 - \frac{\vec{v}_2 \cdot \vec{v}_3}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1 \end{pmatrix}$

So: $\{\vec{v}_1, \vec{v}_3, \vec{w}\}$ – orthogonal eigenbasis

$\Downarrow$ make orthonormal eigenbasis

$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $\vec{u}_3 = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{2} \end{pmatrix}$

So: $\boxed{A = PDP^{-1} \text{ with } D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}}$