

Revision Section #2

Eigenvectors & Eigenvalues

- Definition and the case of triangular matrices
- Linear independence of eigenvectors corresp. to different eigenvalues
- Finding eigenvalues as roots of the characteristic polynomial $p(\lambda) := \det(A - \lambda \cdot I)$
- Finding eigenspace (& its basis) via solving the corresponding homogeneous linear system
- Similar matrices ($A \mapsto P^{-1}AP$ - similarity transformation)

Diagonalization problem: writing A as $A = P \cdot D \cdot P^{-1}$, where P -invertible, D -diagonal

- Find eigenvalues \rightarrow determines D
- Find a basis of eigenspace \rightarrow determines P .

Useful for many applications (in the class, discussed computation of A^k)

Criteria for a diagonalization of A

Generalization to linear transformations $T: V \rightarrow V$

↳ The matrix of a linear transp. w.r.t. a basis of V

Complex eigenvalues

- complex numbers & operations on them
- for a real $n \times n$ matrix A , complex (non-real) eigenvalues come in conjugate pairs

↳ For a 2×2 real matrix A with a complex eig. $\lambda = a - bi$ ($b \neq 0$) and associated eigenvector \vec{v} , we have a presactia,

$A = PCP^{-1}$, where $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $P = (\text{Re } \vec{v} \quad \text{Im } \vec{v})$

→ Application to differential equations $\vec{x}'(t) = A \cdot \vec{x}(t)$ (repeller, attractor, saddle point, spiral point) (1)

↳ Recall how to solve!

Lecture #29

- Inner Product
- Length
- Orthogonality

dot product in \mathbb{R}^n and its properties

- Length $\|\vec{v}\|$, Distance $\text{dist}(\vec{u}, \vec{v})$, Angle b/w \vec{u} & \vec{v}
- Orthogonality
 - ↳ Orthogonal complement of a subspace
 - ↳ $(\text{Row } A)^\perp = \text{Nul } A$, $(\text{Col } A)^\perp = \text{Nul } A^T$

Orthogonal & orthonormal sets

- ↳ decomposition of any $\vec{y} \in W \subseteq \mathbb{R}^n$ w.r.t. an orthogonal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$ of W

$$\vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dots + \frac{\vec{y} \cdot \vec{u}_k}{\vec{u}_k \cdot \vec{u}_k} \vec{u}_k$$
- ↳ orthogonal projection of any $\vec{y} \in \mathbb{R}^n$ on W with an orthogonal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$

$$\text{proj}_W \vec{y} = \text{same formula as above!}$$

Encoding vectors from an orthonormal set as a matrix A , get $A^T \cdot A = I$

those A are called "orthogonal"

- ↳ if A is square, then $A \cdot A^T = I$ as well

Best Approximation Theorem:

- $\hat{y} := \text{proj}_W \vec{y}$ is the point on W closest to \vec{y}
- ↳ equals $A \cdot A^T \cdot \vec{y}$ if columns of A form an orthonormal basis of W

↳ Gram-Schmidt Process to construct orthogonal bases of W (recall what to do when we only have spanning set of W)

↳ QR-factorization: $A = Q \cdot R$

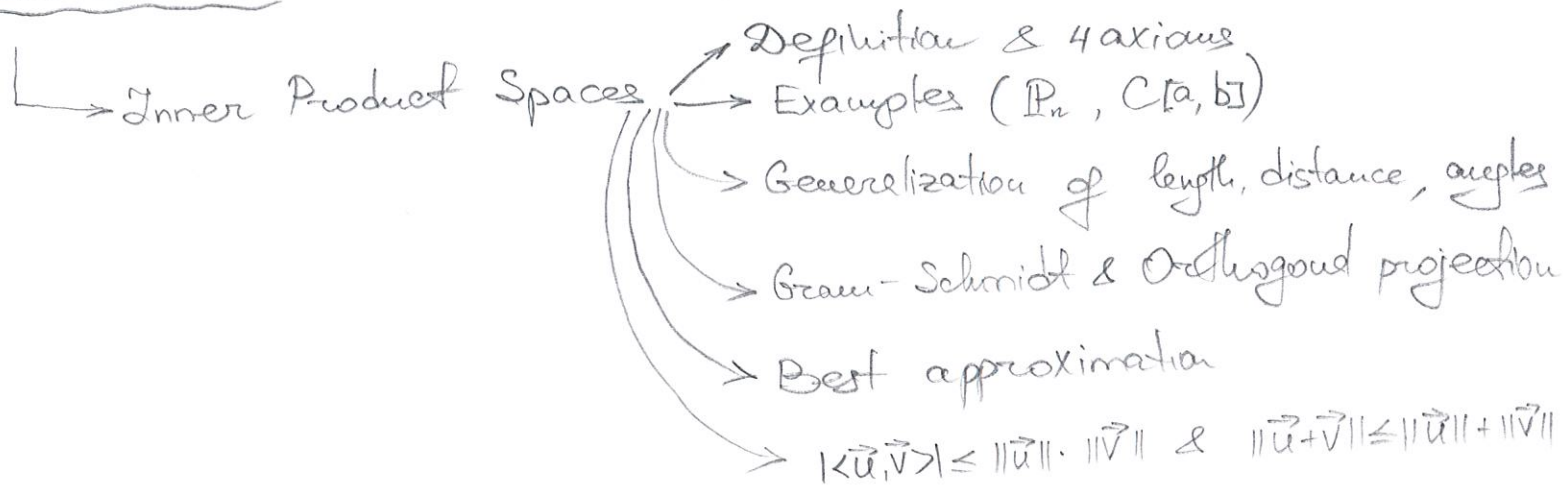
- R ← triangular
- Q ← columns form an orthonormal set
- ↳ Q via Gram-Schmidt for $\text{Col } A$, R via $R = Q^T A$

↳ Least square problems → $\hat{\vec{b}} = \text{proj}_{\text{Col } A} \vec{b}$ - unique, $\hat{\vec{x}}$ - not!

special case when columns of A - orthogonal

- ↳ Find $\hat{\vec{x}}$ by solving $A^T \cdot A \cdot \vec{x} = A^T \cdot \vec{b}$
- ↳ if columns of A - lin. ind, find $\hat{\vec{x}}$ via $\hat{\vec{x}} = R^{-1} \cdot Q^T \cdot \vec{b}$ (2)

Lecture #29



• Symmetric Matrices

$$(A = A^T)$$

→ Basis & dimension

→ Notion of $A \in \text{Mat}_{n \times n}$ being orthogonally diagonalizable

→ Main Result: A -orthog. diag. $\Leftrightarrow A$ -symmetric

↓
Corollary: Any symmetric matrix is diagonalizable (over \mathbb{R})

→ Spectral decomposition (p. 422)

$$A = \lambda_1 \vec{u}_1 \cdot \vec{u}_1^T + \dots + \lambda_n \vec{u}_n \cdot \vec{u}_n^T$$

where λ_i - eigenvalues

\vec{u}_i - orthonormal eigenvectors