HOMEWORK 1 (DUE AUGUST 31)

Disclaimer: *Problems with $*$ refer to such that require some knowledge of complex analysis, manifold theory, or topology. If you have not taken these courses, view this as an opportunity to learn a bit of each (please come to office hours, or you can find most of these results online or in relevant books). Failure to do these problems will* <u>*not*</u> *affect your course grade.*

1. (a) Verify that the two-dimensional sphere S^2 is real analytic and admits a two-chart atlas.

(b) Show that S^2 is also complex analytic and admits a two-chart atlas.

2. (a) Verify that the complex plane $X = \mathbb{C}$ and the open disk $D_r = \{z \in \mathbb{C} | |z| < r\}$ $(r > 0)$ are isomorphic as real analytic manifolds.

(b*) Show that X and D_r are not isomorphic as complex analytic manifolds.

3. (a) Show that an injective regular map of manifolds $X \to Y$ need not be an immersion.

(b) Show that an immersion of manifolds $X \to Y$ need not be injective.

4. (a*) Let $f_1, \ldots, f_m \colon \mathbb{R}^n \to \mathbb{R}$ be C^k $(k \geq 1)$ or real analytic functions, defining a map $f = (f_1, \ldots, f_m) \colon \mathbb{R}^n \to \mathbb{R}^m$. Consider the locus $X \subset \mathbb{R}^n$ of points P such that $f(P) = 0$ and $d_P F$ has rank m (that is, $df_i(P)$ are linearly independent). Verify that X is an $(n-m)$-dimensional topological manifold, and equip it with a natural C^k or real analytic structure.

Hint: Use the implicit function theorem.

(b) Use part (a) to deduce that for any submersion of manifolds $f \colon X \to Y$ and any point $y \in Y$, the preimage $f^{-1}(y)$ is a manifold of dimension $\dim(X) - \dim(Y)$.

5. (a*) Show that the closed interval $[0, 1]$ is connected.

(b) Deduce that a path-connected topological space X is always connected.

(c*) Provide an example of a connected but not path-connected topological space.

(d) Use the fact that an open ball $D_r^n = \{\vec{x} \in \mathbb{R}^n | |\vec{x}| < r\}$ is path-connected to show that a connected manifold X is in fact path-connected (thus, the two notions agree for manifolds).

Hint: For any $x \in X$ show that $U = \{y \in X | \text{there is a path from } x \text{ to } y\}$ is open and closed.

6. Prove that if topological spaces X and Y are connected, then so is $X \times Y$.

7. For which n, is the sphere $S^n = \{\vec{x} \in \mathbb{R}^{n+1} | |\vec{x}| = 1\}$ simply connected?

8*. Prove the *homotopy lifting property of coverings*, as stated in the end of Lecture 2.