HOMEWORK 2 (DUE SEPTEMBER 7)

Disclaimer: *Problems with $*$ refer to such that require some knowledge of manifold theory or topology. If you have not taken these courses, view this as an opportunity to learn a bit of each (please come to office hours, or you can find most of these results online or in books). Failure to do these problems will* not *affect your course grade.*

1*. (a) Verify that the space $\widetilde{X}$ from the beginning of Lecture 3 is indeed simply connected.

(b) Show that for any path-connected covering $p\colon E \to X$, there is a covering $\widetilde{\pi}\colon \widetilde{X} \to E$ such that $p \circ \widetilde{\pi} = \pi\colon \widetilde{X} \to X$.

(c) Deduce that the universal covering is unique, up to an isomorphism.

2. (a) Show that every discrete normal subgroup of a connected Lie group is central.

(b) Deduce that $\mathrm{Ker}(\pi\colon \widetilde{G} \to G)$ is a central subgroup of $\widetilde{G}$ ([Lecture 3, Lemma 2]).

3. Let H be a closed Lie subgroup of a Lie group G.

(a) Show that the closure $\overline{H}$ of H in G is a subgroup of G.

(b) Show that each coset Hx (for $x \in \overline{H}$) is open and dense in $\overline{H}$.

(c) Deduce that $\overline{H} = H$, thus proving that H is closed in G ([Lecture 3, Lemma 3]).

4*. Show that if H is a normal closed Lie subgroup of a Lie group G, then G/H is a Lie group (this is the last part of Proposition 2 from Lecture 3).

5*. (a) If H is a connected closed Lie subgroup of a Lie group G, then $\pi_0(G) = \pi_0(G/H)$.

(b) If both H and G are connected, construct an exact sequence of fundamental groups

$$\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \{1\}\,.$$

6. (a) For any sequence $\underline{d} = (d_1, \ldots, d_k) \in \mathbb{Z}^k$ with $0 < d_1 < d_2 < \cdots < d_k = n$, consider the set $\mathcal{F}_{\underline{d}}(\mathbb{R})$ of partial flags, i.e. sequences of subspaces $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{R}^n$ with $\dim_{\mathbb{R}}(V_i) = d_i$ for all i. Show that the natural action of $GL_n(\mathbb{R})$ on $\mathcal{F}_{\underline{d}}(\mathbb{R})$ is transitive, thus equipping $\mathcal{F}_{\underline{d}}(\mathbb{R})$ with a manifold structure. Identify the fibers of this fiber bundle.

(b) Construct a transitive action of $SU(2)$ on $\mathbb{CP}^1 \simeq S^2$ with stabilizers $U(1) \simeq S^1$, thus recovering the **Hopf fibration** $S^1 \hookrightarrow S^3 \to S^2$.

7. Verify all the properties of exp and log maps stated in Lecture 4.

8. (a) Describe explicitly the Lie algebras of $Sp(2n, \mathbb{K}), O(p,q), SO(p,q), U(p,q), SU(p,q)$.

(b) Compute dimensions of all classical groups.