HOMEWORK 3 (DUE SEPTEMBER 14)

1. Compute π_0 and π_1 (the fundamental group) of the following classical Lie groups:

$$SU(n)\,,\quad U(n)\,,\quad SO(m,\mathbb{R})\,,\quad O(m,\mathbb{R}) \qquad \text{for all} \quad n \geq 1, m \geq 2$$

Hint: Apply [Homework 2, Problem 5] to the fiber bundles from Lecture 4 (and use [Homework 1, Problem 7]) to reduce the problem to the simplest cases $n = 1$ and $m = 2$.

2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Given an open cover $\{U_\alpha\}_{\alpha\in\mathfrak{I}}$ of X and a collection $\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{I}}$ of clutching functions

$$h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

satisfying the consistency conditions

$$(\dagger) \qquad h_{\alpha\beta}(x) = h_{\beta\alpha}(x)^{-1} \quad \text{and} \quad h_{\alpha\beta}(x)\circ h_{\beta\gamma}(x) = h_{\alpha\gamma}(x) \qquad \forall\ \alpha,\beta,\gamma \in \mathfrak{I}$$

construct a $\mathbb{K}$-vector bundle of rank n on X.

(b) Given an atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\mathfrak{I}}$ on an analytic manifold X, with $\varphi_\alpha\colon U_\alpha \to \mathbb{K}^n$, define

$$h_{\alpha\beta}(x) := d_{\varphi_\beta(x)}(\tau_{\alpha\beta}) \in GL_n(\mathbb{K})$$

as the differentials of the corresponding transition maps

$$\tau_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha\cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)\,.$$

Verify ($\dagger$) for $h_{\alpha\beta}$ (completing the construction of the tangent bundle TX from Lecture 5).

(c) Show that a $\mathbb{K}$-vector bundle $\pi\colon E \twoheadrightarrow X$ of rank n is trivial iff there exist n sections $s_1,\ldots,s_n \in \Gamma(X,E)$ such that $\{s_1(x),\ldots,s_n(x)\}$ is a basis of the fiber $\pi^{-1}(x)$ for any $x \in X$.

3. (a) Show that the assignment $V \mapsto V(1)$ gives rise to a vector space isomorphism

$$\big\{\text{biinvariant vector fields on } G\big\} \xrightarrow{\sim} (T_1G)^{Ad\,G} := \big\{x \in T_1G \mid Ad\,g(x) = x\ \forall\, g\in G\big\}$$

(b) For $G = GL_n(\mathbb{R}) \subset End(\mathbb{R}^n)$, let us identify each tangent space $T_gG \simeq End(\mathbb{R}^n)$. Verify that the value of the left-invariant vector field $V_x \in Vect(G)$ at $g \in G$ is given by $(L_g)_*v = gv$, the usual product of matrices. Provide a similar description of the right-invariant vector fields.

4. Let G be a real/complex Lie group. Show that:

(a) If H is a Lie subgroup in G, then $Lie(H)$ is a Lie subalgebra in $Lie(G)$.

(b) If H is a normal closed Lie subgroup in G, then $Lie(H)$ is an ideal in $Lie(G)$.

(c) If H is a closed Lie subgroup in G, both H and G are connected, and $Lie(H)$ is an ideal in $Lie(G)$, then H is normal.

5. Let G be a Lie group with $\mathfrak{g} = Lie(G)$. Define $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Der}(\mathfrak{g})$ via

$$\mathrm{Aut}(\mathfrak{g}) = \{g \in GL(\mathfrak{g}) \mid g[a,b] = [ga, gb] \quad \forall\, a, b \in \mathfrak{g}\}$$
$$\mathrm{Der}(\mathfrak{g}) = \{x \in \mathfrak{gl}(\mathfrak{g}) \mid x[a,b] = [xa, b] + [a, xb] \quad \forall\, a, b \in \mathfrak{g}\}$$

(a*) Show that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with the Lie algebra $\mathrm{Der}(\mathfrak{g})$.

Hint: Use the orbit-stabilizer theorem for Lie groups, see [Lecture 4, Proposition 1].

(b) Verify that $g \mapsto \mathrm{Ad}\, g$ is a homomorphism of Lie groups $G \to \mathrm{Aut}(\mathfrak{g})$ (the automorphisms of the form $\mathrm{Ad}\, g$ are called *inner automorphisms*).

(c) Verify that $x \mapsto \mathrm{ad}\, x$ is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ (the derivations of the form $\mathrm{ad}\, x$ are called *inner derivations*).

(d) Show that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$.

6. (a) Prove that $\mathbb{R}^3$ with the commutator given by cross-product is a Lie algebra.

(b) Construct a Lie algebra isomorphism $\phi\colon \mathfrak{so}_3(\mathbb{R}) \to \mathbb{R}^3$ (see part (a)), which intertwines the standard action $\mathfrak{so}_3(\mathbb{R}) \curvearrowright \mathbb{R}^3$ with the action $\mathbb{R}^3 \curvearrowright \mathbb{R}^3$ given by the cross-product:

$$x \cdot \vec{v} = \phi(x) \times \vec{v} \qquad \forall\, x \in \mathfrak{so}_3(\mathbb{R}), \vec{v} \in \mathbb{R}^3 .$$