HOMEWORK 4 (DUE SEPTEMBER 21)

1. (a) Verify that the formulas for the dual, the direct sum, and the tensor product of representations of a Lie algebra $\mathfrak{g}$, as defined in class, indeed produce $\mathfrak{g}$-representations.

(b) For finite-dimensional $\mathfrak{g}$-modules V, W, identifying $V^* \otimes W \simeq \mathrm{Hom}(V, W)$ as vector spaces, endow $\mathrm{Hom}(V, W)$ with a $\mathfrak{g}$-module structure.

(c) For a $\mathfrak{g}$-module (U, ρ_U), the **space of $\mathfrak{g}$-invariants** is defined as

$$U^{\mathfrak{g}} := \{u \in U \mid \rho_U(x)(u) = 0 \ \forall x \in \mathfrak{g}\}\,.$$

Verify that $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$, where the right-hand side denotes $\mathfrak{g}$-module homomorphisms, while the left-hand side denotes the space of $\mathfrak{g}$-invariants of the module from (b).

(d*) For $\mathfrak{g} = Lie(G)$, derive the formulas for the $\mathfrak{g}$-action on $V \oplus W$, V^*, and $V \otimes W$ from the corresponding constructions for G-modules (spell out the latter ones).

2. (a) Verify that an $\mathfrak{sl}_2$-module is the same as a quadruple $(V; E, H, F)$ of a vector space V with three linear operators $E, F, H \colon V \to V$ satisfying

$$[E, F] = H\,, \quad [H, E] = 2E\,, \quad [H, F] = -2F\,.$$

(b) Conclude that $V = \mathbb{C}[x, y]$ with $E = x\partial_y, H = x\partial_x - y\partial_y, F = y\partial_x$ give rise to $\mathfrak{sl}_2 \curvearrowright V$.

3. Recall the definition of the **character** of any finite-dimensional $\mathfrak{sl}_2$-module (V, ρ_V):

$$\chi_V(z) := tr_V(z^h) = \sum_m \dim V(m)\, z^m$$

where $V(m)$ denotes a generalized eigenspace of $\rho_V(h)$ with eigenvalue $m \in \mathbb{C}$.

(a) Show that $V(m)$ coincides with the m-eigenspace of $\rho_V(h)$ and $V(m) = 0$ unless $m \in \mathbb{Z}$.

(b) Prove the character formulas for the dual, the direct sum, and the tensor product:

$$\chi_{V^*}(z) = \chi_V(z^{-1})\,, \qquad \chi_{V\oplus W}(z) = \chi_V(z) + \chi_W(z)\,, \qquad \chi_{V\otimes W}(z) = \chi_V(z)\chi_W(z)\,.$$

(c) Deduce the Clebsch-Gordan formula by matching the characters of both sides.

4. (a) For a $\mathbb{Z}_{\geq 0}$-filtered associative algebra $\mathcal{A}$ show that if $\mathrm{gr}\,\mathcal{A}$ is a domain, then so is $\mathcal{A}$.

(b) For a linear map $\varphi \colon \mathcal{A} \to \mathcal{B}$ of $\mathbb{Z}_{\geq 0}$-filtered associative algebras satisfying $\varphi(F_k\mathcal{A}) \subset F_k\mathcal{B}$, construct $\mathrm{gr}\,\varphi \colon \mathrm{gr}\,\mathcal{A} \to \mathrm{gr}\,\mathcal{B}$. Show that if $\mathrm{gr}\,\varphi$ is a vector space isomorphism, then so is φ.

5.(a) Verify that the Casimir element $C := fe + ef + h^2/2 \in U(\mathfrak{sl}_2)$ is central.

(b*) Prove that the center $ZU(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$ is a polynomial algebra in C.

6. For any finite-dimensional complex vector space V, verify that V, all its symmetric powers S^nV, and all exterior powers $\Lambda^m V$ $(m \leq \dim V)$ are irreducible representations of $GL(V)$.