HOMEWORK 5 (DUE SEPTEMBER 28)

1. Verify that the symmetrization map $\sigma\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is a $\mathfrak{g}$-module homomorphism.

2. Let V be an n-dimensional vector space over a field $\mathbf{k}$, and $L(V) = \bigoplus_{m\geq 1} L_m(V)$ be the free Lie algebra generated by V with the natural $\mathbb{Z}_{>0}$-grading. Prove that the dimensions of its graded components $d_m(n) := \dim_{\mathbf{k}} L_m(V)$ are uniquely determined by (q–formal variable)

$$\prod_{m\geq 1} (1-q^m)^{d_m(n)} = 1 - nq\,.$$

3. Recall the coproduct map Δ on either $T(\mathfrak{g})$ or $U(\mathfrak{g})$.

(a) Show that an element x is primitive if and only if $\exp(x)$ is group-like.

(b*) Show that if $\mathrm{char}(\mathbf{k}) = 0$, then any primitive element of $U(\mathfrak{g})$ is an element of $\mathfrak{g} \subset U(\mathfrak{g})$.

Hint: Reduce to $S(\mathfrak{g})$ and interpret Δ on $S(V) = \boldsymbol{k}[V^*]$ for an abelian Lie algebra V.

4. (a) Show that any subalgebra and quotient algebra of a solvable Lie algebra are solvable.

(b) Show that any subalgebra and quotient algebra of a nilpotent Lie algebra are nilpotent.

(c) If an ideal I of a Lie algebra $\mathfrak{g}$ and the quotient $\mathfrak{g}/I$ are solvable, is $\mathfrak{g}$ solvable?

(d) If an ideal I of a Lie algebra $\mathfrak{g}$ and the quotient $\mathfrak{g}/I$ are nilpotent, is $\mathfrak{g}$ nilpotent?

5. (a) Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra with basis x, y, z and commutation relations

$$[x,y] = z\,, \qquad [x,z] = 0\,, \qquad [y,z] = 0\,,$$

called the **Heisenberg algebra**. Construct explicitly the connected, simply-connected Lie group corresponding to $\mathfrak{g}$, and verify without using the Campbell–Hausdorff formula that $\exp(tx)\exp(sy) = \exp(tsz)\exp(sy)exp(tx)$.

(b) Generalize the previous part to the Lie algebra $\mathfrak{g} = V \oplus \mathbb{R}z$, where V is a real vector space with a non-degenerate skew-symmetric form $\omega\colon V \otimes V \to \mathbb{R}$ and the commutation relations

$$[u,v] = \omega(u,v)z\,, \qquad [z,v] = 0 \qquad \forall\ u,v \in V\,.$$

6. For a $\mathfrak{g}$-module (V,ρ), the space of **coinvaraints** is defined as

$$V_{\mathfrak{g}} = V/\mathfrak{g}V \qquad \text{where} \quad \mathfrak{g}V = \mathrm{span}\{\rho(x)v \,|\, x \in \mathfrak{g}, v \in V\}\,.$$

(a) Prove that for completely reducible V, the composition $V^{\mathfrak{g}} \to V \to V_{\mathfrak{g}}$ is an isomorphism.

(b) Show that in general, it is not so.

7. Given a Lie algebra $\mathfrak{h}$ and another Lie algebra $\mathfrak{g}$ acting on $\mathfrak{h}$ by derivations, one defines the **semidirect product** Lie algebra $\mathfrak{g} \ltimes \mathfrak{h}$ which is $\mathfrak{g} \oplus \mathfrak{h}$ as a vector space with the commutator

$$[(x_1,h_1),(x_2,h_2)] = ([x_1,x_2], [h_1,h_2] + x_1(h_2) - x_2(h_1))\,.$$

Verify this indeed produces a Lie algebra.