HOMEWORK 6 (DUE OCTOBER 5)

1. For $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$, set $n := n_1 + \ldots + n_k$, and consider the **parabolic subalgebra**

$$\mathfrak{g} = \big\{ A \in \mathfrak{gl}(n) \,|\, A_{ij} = 0 \text{ for } j \le n_1 + \ldots + n_s < i \text{ and any } 1 \le s < k \big\}$$

consisting of block upper triangular matrices with diagonal blocks of size $n_1 \times n_1, \ldots, n_k \times n_k$.

(a) Find the radical $\mathrm{rad}(\mathfrak{g})$.

(b) Provide an example of the Levi decomposition for this $\mathfrak{g}$.

2. (a) Given a finite-dimensional $\mathfrak{g}$-module (V, ρ), verify that the following defines an action of $\mathfrak{g}$ on the space Bil_V of all bilinear maps $V \times V \to \mathsf{k}$:

$$(xB)(v, w) = -B(\rho(x)v, w) - B(v, \rho(x)w) \qquad \forall\, x \in \mathfrak{g}\,,\ v, w \in V$$

(b) Verify that $B \in \mathrm{Bil}_V^{\mathfrak{g}}$ iff the corresponding linear map is a $\mathfrak{g}$-module homomorphism:

$$V \to V^* \quad \text{given by} \quad v \mapsto B(v, -)$$

(c) If V is an irreducible $\mathfrak{g}$-module, show that $\dim \mathrm{Bil}_V^{\mathfrak{g}} \le 1$.

(d) If $(V, \rho) = (\mathfrak{g}, \mathrm{ad})$ and $I \subset \mathfrak{g}$ is an ideal, then $I^\perp = \{x \in \mathfrak{g} | B(x, y) = 0\ \forall\, y \in I\}$ is an ideal (here, B is assumed to be a $\mathfrak{g}$-invariant form on $\mathfrak{g}$).

3. (a) For any filtration $0 = F_0 V \subset F_1 V \subset \cdots \subset F_N V = V$ by $\mathfrak{g}$-submodules verify that

$$B_V = \sum_{1 \le k \le N} B_{F_k V / F_{k-1} V} \colon \mathfrak{g} \times \mathfrak{g} \to \mathsf{k}$$

(b) For an ideal $\mathfrak{a} \subset \mathfrak{g}$, verify the equality $K^{\mathfrak{a}}(x, y) = K^{\mathfrak{g}}(x, y)$ for any $x, y \in \mathfrak{a}$.

4. (a) Show that $\mathfrak{sp}_{2n}, \mathfrak{u}_n, \mathfrak{su}_n$ are reductive (finishing the proof of [Lecture 12, Theorem 1]).

(b) Verify that $\mathfrak{sl}_n(\mathsf{k}), \mathfrak{so}_n(\mathsf{k}), \mathfrak{sp}_{2n}(\mathsf{k}), \mathfrak{su}_n$ are semisimple.

(c) Verify that $\mathfrak{gl}_n(\mathsf{k}) = \mathsf{k} \cdot \mathrm{Id} \oplus \mathfrak{sl}_n(\mathsf{k})$ and $\mathfrak{u}_n = i\mathbb{R} \cdot \mathrm{Id} \oplus \mathfrak{su}_n$.

5. Let V be a finite-dimensional complex vector space.

(a) Verify that a linear operator $A \colon V \to V$ is semisimple iff A is diagonalizable.

(b) Show that if $A \colon V \to V$ is semisimple and a subspace $V' \subset V$ is A-stable, i.e. $A(V') \subset V'$, then the corresponding linear operators $V' \to V'$ and $V/V' \to V/V'$ are also semisimple.

(c) Show that the sum of two commuting semisimple operators $V \to V$ is also semisimple.

(d) Show that the sum of two commuting nilpotent operators $V \to V$ is also nilpotent.

6. (a) Verify that if $A \colon V \to V$ is nilpotent, then $\mathrm{ad}(A) \colon \mathrm{End}(V) \to \mathrm{End}(V)$ is nilpotent.

(b) Prove part (c) of the Jordan decomposition ([Lecture 12, Theorem 4(c)]).